Exercises 1–3 are "compulsory" and 4–6 "bonus problems" (marked always with † or †††).

Notice that there are more bonus exercises this week than previously. If time forbids presenting of all the solutions in the exercise session, the exercises 5–6 can be postponed to the following week!

Exercise 1. Recall the Borel–Cantelli lemma (by reading Durrett's book, say, if necessary). Show that for any sequence of random variables X_n, there exists a sequence of constants $l_n > 0$ such that

$$\mathsf{P}\left[\lim_{n\to\infty} \frac{X_n}{l_n} = 0\right] = 1.$$

(We say that X_n/l_n converges to zero almost surely.)

Exercise 2 (A construction of Brownian motion on $[0,1]$)**.** Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space with a countably infinite set $(\xi(t))_{t\in\mathcal{D}}$ of independent $N(0,1)$ random variables, where $\mathcal{D} = \bigcup \mathcal{D}_n$ and $\mathcal{D}_n = \{k\,2^{-n} : k = 0,1,2,\dots,2^n\}$ are the sets of *dyadic* rationals. (Such probability space can be constructed, for example, as a product space.)

(a) We are first going to construct a process $B(t)$, $t \in \mathcal{D}$, such that it agrees with the distribution of Brownian motion on the dyadic points.

On $\mathcal{D}_0 = \{0,1\}$, define $B(0) = 0$ and $B(1) = c_0\xi(1)$. Define then recursively $B(t)$ on $t \in \mathcal{D}_n \smallsetminus \mathcal{D}_{n-1}$ for $n \geq 1$, by setting

$$B(t) = a_n\, B(t - 2^{-n}) + b_n\, B(t + 2^{-n}) + c_n\, \xi(t)$$

for any $t \in \mathcal{D}_n \smallsetminus \mathcal{D}_{n-1}$. Find the real numbers c_0, a_n, b_n, c_n, $n \geq 1$ such that $B(1)$ has the correct distribution (Brownian motion at time 1) and the conditional distribution of $B(t), t \in \mathcal{D}_n \smallsetminus \mathcal{D}_{n-1}$, given $B(t), t \in \mathcal{D}_{n-1}$, is the correct one (which is the Brownian bridge distribution from Exercise 2(b) of Problem Sheet 2).

(b) Now we extend the process $B(t)$ to the whole interval $[0,1]$.

Let $B|_{\mathcal{D}_n}$ be the restriction of B to $\mathcal{D}_n$ and define $B^{(n)}(t)$ as the extension of $B|_{\mathcal{D}_n}$ by linear interpolation to $[0,1]$. Define piecewise linear processes $Z^{(0)} = B^{(0)}$ and $Z^{(n)} = B^{(n)} - B^{(n-1)}$, $n \geq 1$. Show that

$$\sum_{n=0}^{\infty} \mathsf{P}[\text{for some } t \in \mathcal{D}_n,\ |\xi(t)| \geq c\sqrt{n}] < \infty$$

when $c > \sqrt{2\log 2}$. Use Borel–Cantelli lemma to conclude that the sum $\sum_{n=0}^{\infty} \|Z^{(n)}\|_\infty$ is almost surely finite and that almost surely the series

$$B(t) = \sum_{n=0}^{\infty} Z^{(n)}(t)$$

converges uniformly in $[0,1]$. Here $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.

Exercise 3. Let $(B_t)_{t\in[0,1]}$ be the process constructed in the previous exercise. Prove that the increments of B_t are independent and that $B_{s+t} - B_s$ is normally distributed with mean 0 and variance t.

Turn page!

† **Exercise 4.** Let $f \in \mathcal{L}^2$ be simple and bounded. Define $X_t(\omega) = \int_0^t f \, \mathrm{d}B_s(\omega)$ and $\langle X \rangle_t(\omega) = \int_0^t f(s,\omega)^2 \, \mathrm{d}s$. Show that $X_t^2 - \langle X \rangle_t$ is a martingale. Show also that $\langle X \rangle$ is the quadratic variation process $V_X^{(2)}$ in the sense of the definition in Section 2.1 of the lecture notes.

† **Exercise 5** (Doob's submartingale inequality in discrete time)**.** Let $(M_n)_{n \in \mathbb{Z}_+}$ be a non-negative discrete-time *submartingale*[1] with respect to a (discrete-time) filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$.

(a) Show that

$$\mathsf{E}[M_m \mathbb{1}_A] \leq \mathsf{E}[M_n \mathbb{1}_A]$$

for all $A \in \mathcal{F}_m$ and all $n \geq m$.

Hint. The first step is to use the definition of the conditional expected value $\mathsf{E}[M_{m+1} | \mathcal{F}_m]$ or $\mathsf{E}[M_n | \mathcal{F}_m]$ and integrate over the measurable set A.

(b) Let $\tau = \min\{m : M_m \geq \lambda\}$. Show that

$$\lambda \mathbb{1}_{\tau \leq n} \leq \sum_{m=0}^{n} M_m \mathbb{1}_{\{\tau = m\}}$$

(c) Show that

$$\mathsf{P}\left[\max_{0 \leq m \leq n} M_m \geq \lambda\right] \leq \frac{1}{\lambda} \mathsf{E}[M_n]$$

† **Exercise 6** (Doob's submartingale inequality in continuous time)**.** Let $(M_t)_{t \in \mathbb{R}_+}$ be a continuous non-negative (continuous-time) submartingale with respect to a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Use the previous exercise to show that for any $T > 0$

$$\mathsf{P}\left[\sup_{0 \leq t \leq T} M_t \geq \lambda\right] \leq \frac{1}{\lambda} \mathsf{E}[M_T].$$

Explain how Doob's martingale inequality of Appendix A follows from this result.

Hint. Set $\pi_n = \{k \, T \, 2^{-n} : k = 0, 1, 2, \ldots, 2^n\}$ for any $n \in \mathbb{N}$ and consider first $\sup_{t \in \pi_n} M_t$.

[1] See Appendix A for the definitions of discrete-time and continuous-time martingale, submartingale and supermartingale.