Exercises 1–3 are "compulsory" and 4–5 "bonus problems" (marked always with † or †††).

Exercise 1. (a) Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. Show that $\mathsf{Cov}[B_t, B_s] = \min\{t, s\}$.

Hint. For $s < t$, use $B_t = B_s + (B_t - B_s)$.

(b) Let $0 \le t < s$. Show that for any $\lambda \in (0, 1)$, the random variable

$$Y_\lambda = B_{\lambda t+(1-\lambda)s} - \lambda B_t - (1-\lambda)B_s$$

is independent from $\sigma(B_u\,,\, 0 \le u \le t)$ and $\sigma(B_u\,,\, u \ge s)$.

Hint. It is sufficient to check independence of Y_λ and single B_u for any $u \notin (t, s)$. Use properties of multivariate normal distributions.

(c) Show that $\mathsf{E}[Y_\lambda^2] = \lambda(1-\lambda)\,|s-t|$.

Exercise 2. (a) Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion. For any $0 < t < s$, find the conditional density of B_t given B_s in the sense of Exercise 5 below. Illustrate this distribution by drawing its mean and standard deviation as functions of t.

(b) Conclude from (a) and Exercise 1 that for any $0 \le r < s$ and $x, y \in \mathbb{R}$, conditionally on $B_r = x$ and $B_s = y$ the processes $(B_t)_{t\in[0,r)}$, $(B_t)_{t\in(r,s)}$ and $(B_t)_{t\in(s,\infty)}$ are independent. Lastly, when $0 = s_0 < s_1 < \ldots < s_n$ and $x_k \in \mathbb{R}$ and $t_k \in (s_{k-1}, s_k)$ for $k = 1, 2, \ldots, n$, describe the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ given $B_{s_1} = x_1, B_{s_2} = x_2, \ldots, B_{s_n} = x_n$.

Exercise 3. (a) Let X be a Gaussian random variable with mean 0 and variance 1. Show that for any $x > 0$,

$$\frac{x}{\sqrt{2\pi}(1+x^2)}\exp\left(-\frac{x^2}{2}\right) \le \mathsf{P}\big[X \ge x\big] \le \frac{1}{\sqrt{2\pi}\,x}\exp\left(-\frac{x^2}{2}\right).$$

(b) Let $X_n \sim N(\mu_n, \sigma_n^2)$ be a sequence of Gaussian random variables such that $X_n \to X$ almost surely and $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ as $n \to \infty$. Show that $X \sim N(\mu, \sigma^2)$. (Here, as usual, $X \sim N(\mu, \sigma^2)$ means that X is distributed normally with mean μ and variance σ^2.)

Hint. Recall different ways to characterize a probability distribution.

† **Exercise 4.** (a) Let $\mathcal{A}$ be a σ-algebra such that for all $A \in \mathcal{A}$, $\mathsf{P}[A] = 0$ or 1. Show that $\mathsf{E}[X|\mathcal{A}] = \mathsf{E}[X]$ for any $X \in L^1$.

(b) Let $\Omega_1, \Omega_2, \ldots$ be a finite or countably infinite partition of Ω into $\mathcal{F}$-measurable sets, i.e., $\Omega_j \cap \Omega_k = \emptyset$ when $j \ne k$ and $\bigcup_{k=1}^\infty \Omega_k = \Omega$. Assume that each Ω_k has positive probability. Let $\mathcal{G}$ be the σ-algebra generated by $\Omega_1, \Omega_2, \ldots$ Show that

$$\mathsf{E}[X|\mathcal{G}] = \frac{\mathsf{E}[X;\Omega_k]}{\mathsf{P}[\Omega_k]} \quad \text{on } \Omega_k.$$

Here we use the standard notation $\mathsf{E}[X;E] = \int_E X\,\mathrm{d}\mathsf{P}$.

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† **Exercise 5.** Let X and Y be two random variables that have a joint density $f(x,y)$ in the sense that for any bounded, Borel (measurable) function ϕ on $\mathbb{R}^2$

$$\mathsf{E}[\phi(X,Y)] = \int_{\mathbb{R}^2} \phi(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Define the marginal density of Y by

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x$$

and let

$$f(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & \text{, if } f_Y(y) > 0 \\ 0 & \text{, if } f_Y(y) = 0. \end{cases}$$

We call the quantity $f(x|y)$ the *conditional density of X given $Y = y$*.

(a) Show that $f(x|y) f_Y(y) = f(x,y)$ for almost every (x,y) with respect to the Lebesgue measure on $\mathbb{R}^2$.

Hint: Prove first that the Lebesgue measure of $\{x : f(x|y) f_Y(y) \neq f(x,y)\}$ is zero for each y and then use Fubini's theorem.

(b) Show that $f(x|y)$ can be seen as the conditional density of X given $Y = y$ in the sense that

$$\mathsf{E}[h(X)|Y](\omega) = \int_{\mathbb{R}} h(x) f(x|Y(\omega))\mathrm{d}x$$

for any bounded Borel function h on $\mathbb{R}$.