Exercises 1–3 are "compulsory" and 4 is a "bonus problem"

Exercise 1. Continue the setup of Exercise 3 in Problem sheet 9. That is, let $\nu = \pm 1$, $\kappa > 0$ and $h_t(z)$ be the solution of the differential equation

$$\partial_t h_t(z) = \nu \frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z$$

with $W_t = -\sqrt{\kappa} B_t$ where $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion with respect to a σ-algebra $\mathcal{F}_t$. Fix $z_0 \in \mathbb{H}$ and set $X_t = \operatorname{Re} h_t(z_0) - W_t$, $Y_t = \operatorname{Im} h_t(z_0)$ and $Z_t = X_t + \mathrm{i} Y_t$. Verify all the following formulas

$$\mathrm{d}X_t = 2\nu \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \partial_t Y_t = -2\nu \frac{Y_t}{X_t^2 + Y_t^2}, \qquad \partial_t \frac{|h_t'(z)|}{Y_t} = 4\nu \frac{|h_t'(z)|}{Y_t} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2}$$

$$\mathrm{d} \arg Z_t = (\kappa - 4\nu) \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2} \mathrm{d}t - \sqrt{\kappa} \frac{Y_t}{X_t^2 + Y_t^2} \mathrm{d}B_t,$$

$$\mathrm{d} \log |Z_t| = -\frac{1}{2}(\kappa - 4\nu) \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2} \mathrm{d}t + \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}B_t,$$

$$\mathrm{d} \sin \arg Z_t = (\sin \arg Z_t) \left[\frac{(\kappa - 4\nu) X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2 + Y_t^2)^2} \mathrm{d}t - \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2} \mathrm{d}B_t \right]$$

Exercise 2. Show using the Koebe distortion theorem that there exists constants C and r such that for any conformal map $f : \mathbb{H} \to \mathbb{C}$ and for any $x \in \mathbb{R}$, $y > 0$ and $1/2 \leq s \leq 2$

$$C^{-1} |f'(iy)| \leq |f'(isy)| \leq C |f'(iy)|,$$

$$C^{-1} (1 + x^2)^{-r} |f'(iy)| \leq |f'(y(x + i))| \leq C (1 + x^2)^r |f'(iy)|.$$

What is the value of r that you get from the Koebe distortion theorem?

Exercise 3. (a) For a Loewner chain g_t, let $f_t = g_t^{-1}$. By differentiating the Loewner equation of f_t with respect to z, find a differential equation for $f_t'(z)$. Show that for $x \in \mathbb{R}$, $y > 0$

$$|\partial_t f_t'(x + iy)| \leq \frac{2|f_t''(x + iy)|}{y} + \frac{2|f_t'(x + iy)|}{y^2}.$$

(b) Show using the special case $|a_2| \leq 2$ of the Bieberbach–de Branges theorem that there is a constant $c > 0$ such that

$$|f''(z)| \leq \frac{c}{\operatorname{Im} z} |f'(z)|$$

for any $f : \mathbb{H} \to \mathbb{C}$ conformal and for any $z \in \mathbb{H}$.

(c) Show that there are constants c_1, c_2, c_3 such that following holds for any Loewner chain: for any $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $y > 0$

$$|\partial_t f_t'(x + iy)| \leq \frac{c_1 |f_t'(x + iy)|}{y^2}$$

and if $0 \leq s \leq y^2$ then

$$|f_{t+s}'(x + iy)| \leq c_2 |f_t'(x + iy)|, \qquad |f_{t+s}(x + iy) - f_t(x + iy)| \leq c_3 \, y \, |f_t'(x + iy)|.$$

Turn page!

† **Exercise 4** (Transformation between Loewner chains in $\mathbb{H}$ and $\mathbb{D}$). Consider an $\mathbb{H}$-hull $K_\delta = [0, \mathrm{i}\delta]$ and a $\mathbb{D}$-hull $\tilde{K}_{\tilde{\delta}} = [1 - \tilde{\delta}, 1]$ where $\delta > 0$ and $0 < \tilde{\delta} < 1$.

(a) Calculate the $\mathbb{H}$-capacity of K_δ and the $\mathbb{D}$-capacity of $\tilde{K}_{\tilde{\delta}}$.

(b) Fix a Möbius map ψ from $\mathbb{H}$ onto $\mathbb{D}$. Calculate c in $\mathrm{cap}_{\mathbb{H}}(K_\delta)/\mathrm{cap}_{\mathbb{D}}(\psi(K_\delta)) = c + \mathcal{O}(\delta)$. What does this tell about transforming Loewner chains from $\mathbb{H}$ to $\mathbb{D}$? Consider in your answer also how a length element around 0 transforms as well as a Brownian motion over a short time interval.