Exercises 1–3 are "compulsory."

Exercise 1. (a) Consider a strip $S_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. Show that for $c \in \mathbb{R}$

$$\phi_c : z \mapsto \log(z - c)$$

is a conformal map from $\mathbb{H}$ onto S_π. Here we use the choice $\operatorname{Im} \log z \in [0, \pi]$, when $z \in \overline{\mathbb{H}} \setminus \{0\}$, for the branch of log. Show that the image of 0 is in $\mathbb{R}$ if and only if $c < 0$.

(b) Let $K \subset \overline{\mathbb{H}}$ be a hull. Let $x \in \mathbb{R} \setminus K$ and $x' = g_K(x)$. Show that

$$G = \phi_{x'} \circ g_K \circ \phi_x^{-1}$$

satisfies

$$G(z) = z - 2a_0 + o(1),\ z \to -\infty, \qquad G(z) = z + o(1),\ z \to +\infty.$$

Find a_0 in terms of g_K and x. Is a_0 positive?

(c) Show that the conformal onto maps $f : S_\pi \to S_\pi$ with $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$ are of the form

$$f(z) = z + C$$

where $C \in \mathbb{R}$ is a constant.

(d) Let's call a compact $K \subset \overline{S_\pi}$ a *s-hull* or a *hull in the strip* if $S_\pi \setminus K$ is simply connected. Show that for any s-hull K, there exists a unique conformal map G_K from $S_\pi \setminus K$ onto S_π such that

$$G_K(z) = z - a_0 + o(1),\ z \to -\infty, \qquad G_K(z) = z + a_0 + o(1),\ z \to +\infty \tag{1}$$

for some constant $a_0 \in \mathbb{R}$. The constant a_0 could be called the *s-capacity* or the *strip capasity* of K. Show that it is additive when mappings with this form of expansion are composed.

Exercise 2 (Chordal SLE(κ) in the strip S_π)**.**

(a) Let $c < 0$. Let $(g_t)_{t\in\mathbb{R}_+}$ be a Loewner chain (in $\mathbb{H}$) with the driving term $(W_t)_{t\in\mathbb{R}_+}$. Motivated by the previous exercise define a time-change $\sigma(s)$ (which depends on c and $(W_t)_{t\in\mathbb{R}_+}$) such that

$$-\frac{1}{2} \log g'_{\sigma(s)}(c) = s$$

and define

$$G_s(z) = \log\left[g_{\sigma(s)}\left(c + |c|e^z\right) - g_{\sigma(s)}\left(c\right)\right] + s - \log|c|.$$

Show that G_s is a conformal map from $S_\pi \setminus \tilde{K}_s$ onto S_π normalized as in (1) where $(\tilde{K}_S)_{s\in\mathbb{R}_+}$ is a growing family of s-hulls and $(G_s)_{s\in\mathbb{R}_+}$ satisfies the Loewner equation of S_π, i.e.,

$$\partial_s G_s(z) = \coth \frac{G_s(z) - \tilde{W}_s}{2}, \qquad G_0(z) = z \tag{2}$$

where $\tilde{W}_s = \log(W_{\sigma(s)} - g_{\sigma(s)}(c)) + s - \log|c|$.

(b) Show that when g_t is SLE(κ), that is, $W_t = \sqrt{\kappa} B_t$, then $\tilde{W}_s$ is a Brownian motion with (linear) drift.

Turn page!

Exercise 3. Reverse the calculations of the previous problems: Suppose that G_s satisfies (2) with a driving process $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$ (note that we have general $\kappa \geq 0$ and $\alpha \in \mathbb{R}$ here). Compose G_s (from both sides) with suitably chosen conformal transformations, so that, G_s is transformed to a hydrodynamically normalized conformal map of $\mathbb{H}$ and $+\infty$ is mapped to ∞, and find the correct time-change so that G_s is transformed to g_t which satisfies the Loewner equation of $\mathbb{H}$ with a driving process W_t. Is g_t well-defined for all t?

Hint. Write first a general form of the transformation leaving free parameters which you then fix so that the resulting map is a hydrodynamically normalized conformal map. Also, the full answer to this exercise should show that W_t satisfies the following system of stochastic differential equations:

$$\mathrm{d}W_t = \sqrt{\kappa}\mathrm{d}B_t + \frac{\rho}{W_t - C_t}\mathrm{d}t$$
$$\mathrm{d}C_t = \frac{2}{C_t - W_t}\mathrm{d}t$$

where $\rho = \rho(\kappa, \alpha) \in \mathbb{R}$ is a constant. Notice that the equation for $\mathrm{d}C_t$ is the Loewner equation of $\mathbb{H}$.