Exercises 1–3 are "compulsory" and 4 is a "bonus problem" (marked always with † or †††).

Notice exceptional time and place, Monday 18.4. 14-16 room CK111.

Exercise 1. (a) As a motivation consider the random curve γ of Schramm's principle with arbitrary parametrization in the domain $(\mathbb{H}, 0, \infty)$. Let $r > 0$ and $\tau = \frac{1}{2}\mathrm{cap}_{\mathbb{H}}(\gamma[0,\sigma])$, where σ is the first time that satisfies $|\gamma(\sigma)| = r$. Let $T_0 = \lim_{t\to\infty} \frac{1}{2}\mathrm{cap}_{\mathbb{H}}(\gamma[0,t])$. Write $T_0 = \tau + T_1$. Explain based on CI and DMP why T_0 and T_1 are equal in distribution and T_1 and τ are independent.

(b) Show that if τ, T_0, T_1 are $[0, +\infty]$-valued random variables, τ is positive with positive probability, T_1 and τ are independent, T_0 and T_1 are equal in distribution and $T_0 = \tau + T_1$, then T_0 and T_1 are almost surely equal to $+\infty$.

Hint. You can use the Laplace transform (a.k.a. the moment-generating function) to solve this problem.

Exercise 2. (a) Let $\nu \in \mathbb{C}$ with $|\nu| = 1$ and let $0 < \alpha < 1$. Show that $z \mapsto \nu z^\alpha$ defines a conformal conformal map from $\mathbb{H}$ to $\mathbb{C}$ (for any continuously chosen branch). Find $\nu = \nu(\alpha)$ such that the image domain is symmetric with respect to the y-axis and lies in $\mathbb{H}$.

(b) Let $h(z) = \mathrm{Im}(\nu z^\alpha)$ where $\nu = \nu(\alpha)$ is as above. Show that there is a constant $C = C(\alpha) \geq 1$ such that $C^{-1}|z|^\alpha \leq h(z) \leq C|z|^\alpha$ for all $z \in \overline{\mathbb{H}}$.

Exercise 3 (Almost surely $\tau(z) < \infty$ when $\kappa > 4$)**.**

(a) Consider SLE(κ) with $\kappa > 4$ and let $\alpha = \alpha(\kappa) = 1 - 4/\kappa$. Show that the real and imaginary parts of Z_t^α are local martingales where $Z_t = g_t(z) - W_t$, $z \in \mathbb{H}$, and $W_t = -\sqrt{\kappa} B_t$. Conclude that $h(Z_t)$ is a local martingale.

(b) For any $R > 0$, define $\sigma_R = \tau(z) \wedge \inf\{t \in [0, \tau(z)) : |Z_t| = R\}$. We make the assumption that $\sigma_R < \infty$ almost surely. (See the article Rohde&Schramm, Basic properties of SLE, Lemma 6.5., for the proof of this fact.) What is the geometric description of σ_R, that is, it is the exit time of Z_t from which set? What are the possible values of Z_{σ_R}?

Let h be as in Exercise 2. Show that $h(Z_{t\wedge\sigma_R})$ is a martingale and show that there exist a constant $\tilde{C} = \tilde{C}(\kappa) \geq 1$ such that

$$\tilde{C}^{-1}\left(\frac{|z|}{R}\right)^\alpha \leq \mathsf{P}\left[|Z_{\sigma_R}| = R\right] \leq \tilde{C}\left(\frac{|z|}{R}\right)^\alpha.$$

Hint. Use the optional stopping theorem.

(c) Deduce that for $\kappa > 4$ and for any $z \in \mathbb{H}$, almost surely $\tau(z) < \infty$.

Turn page!

† **Exercise 4** (Girsanov transformation for random walk).

Let $\Omega = \{\omega : \mathbb{Z}_+ \to \mathbb{Z} : \omega(0) = 0\}$. Let $X_n(\omega) = \omega(n)$, $n \in \mathbb{Z}_+$, be the coordinate maps and let $\mathcal{F}$ be the σ-algebra generated by X_n, $n \in \mathbb{Z}_+$. The space $(\Omega, \mathcal{F})$ is the canonical space for $\mathbb{Z}$-valued discrete-time stochastic processes. Suppose that there is defined a family of probability measures P_p, $0 < p < 1$, on $(\Omega, \mathcal{F})$ such that for any P_p the increments $\xi_n = X_n - X_{n-1}$, $n \in \mathbb{N}$, are independent and identically distributed as

$$\mathsf{P}_p[\xi_n = 1] = p, \qquad \mathsf{P}_p[\xi_n = -1] = 1 - p.$$

Define a filtration $\mathcal{F}_n = \sigma(X_0, \dots, X_n) = \sigma(\xi_1, \dots, \xi_n) \subset \mathcal{F}$.

(a) Let $\mathsf{P}|_{\mathcal{A}}$ be the restriction of a probability measure P on a σ-algebra $\mathcal{A}$. Find an explicit formula for the Radon-Nikodym derivative

$$M_n = \frac{\mathrm{d}\mathsf{P}_p|_{\mathcal{F}_n}}{\mathrm{d}\mathsf{P}_{1/2}|_{\mathcal{F}_n}}$$

in terms of n and X_n.

Hint. $\mathsf{P}_p|_{\mathcal{F}_n}$ is a probability measure on a finite set and it is given by the finite collection of probabilities of singleton sets. Calculate M_n as a ratio of those numbers.

(b) Show that M_n is a martingale for the probability measure $\mathsf{P}_{1/2}$ and for the filtration $(\mathcal{F}_n)_{t \in \mathbb{Z}_+}$.

(c) Use the notation $M_{p,n}$ for the above quantity to highlight the dependency on p. Let $c \in \mathbb{R}$. For each $t \in \mathbb{R}_+$ and $0 < \varepsilon < |c|^{-1}$, define

$$n^{(\varepsilon)}(t) = \lfloor \varepsilon^{-2} t \rfloor, \qquad Y_t^{(\varepsilon)} = \varepsilon X_{n^{(\varepsilon)}(t)}, \qquad p^{(\varepsilon)} = \frac{1}{2}(1 + c\varepsilon)$$

where $\lfloor x \rfloor$ is the largest integer less or equal to the real number x. Find the limit of $M_t^{(\varepsilon)} := M_{p^{(\varepsilon)}, n^{(\varepsilon)}(t)}$ as $\varepsilon \searrow 0$ in terms of $t, Y_t = \lim Y_t^{(\varepsilon)}$ and c.

Note. The scaled simple random walk $Y_t^{(\varepsilon)}$ converges as $\varepsilon \to 0$ in distribution to a Brownian motion in the topology of locally uniform convergence by Donsker's theorem.