Chapter 6
Regularity and convergence of random curves

6.1 Continuity properties of the Loewner map

In this section we will switch to the following notation for Loewner chains.

$$W(t), \quad \gamma(t), \quad g(t,z), \text{ etc.} \quad \text{and} \quad W_n(t), \quad \gamma_n(t), \quad g_n(t,z), \text{ etc.} \tag{6.1}$$

This allows us to denote, for instance, a sequence of driving terms by $(W_n(t))_{t\in[0,T_n)}$. Let us also use the notation

$$F(t,y) = f(t, W_t + \mathrm{i}y). \tag{6.2}$$

The path $y \mapsto F(t,y)$, $y > 0$ is the shortest path in a conformal sense between the point ∞ and the "tip" of K_t.

The variable t takes values in $[0,T)$ for $W(t), \gamma(t), g(t,z)$, etc. and in $[0,T_n)$ for $W_n(t), \gamma_n(t), g_n(t,z)$, etc. The variables T and T_n can be finite or infinite. Since we often consider uniform convergence in compact sets of the time variable we often restrict to $t \in [0,T']$ and consider any T' which is finite and less than T or T_n.

The rest of this section will be added soon!

6.2 Continuity of SLE(κ)

6.2.1 Existence of the trace for Loewner chains

The following lemma tells that certain time instances in a Loewner chain are tame in the sense that $\lim_{y\to 0} F_t(y)$ exists.

Lemma 6.5. *Let $z_0 \in \overline{\mathbb{H}} \setminus \{W_0\}$, $0 < r < |z_0 - W_0|$ and $B = B(z_0, r) \cap \mathbb{H}$. Suppose $t > 0$ is such that $K_t \cap \overline{B}$ is non-empty and $K_s \cap \overline{B}$ is empty for each $s \in [0,t)$. Suppose also that $\overline{B} \setminus K_t$ is non-empty, that is, $\overline{B}$ is hit, but not swallowed by K_t. Then there exists $z_1 \in \partial B$ such that $K_t \cap \overline{B} = \{z_1\}$ and moreover, $z_1 = \lim_{y\to 0} F_t(y)$.*

Proof. Let z_0, r, B and t be as stated above. Then by the assumptions $(\partial K_t) \cap \overline{B} \neq \emptyset$. Let $z_1 \in (\partial K_t) \cap \overline{B}$. By the Carathéodory kernel convergence theorem there exists $w_n \in \partial K_{s_n}$ such that s_n increases to t and w_n tends to z_1 as $n \to \infty$. Since $w_n \in \mathbb{C} \setminus \overline{B}$ for all n, it follows that $z_1 \in \partial B$. Hence $(K_t \cap \overline{B}) \subset \partial B$.

Since the line segment from z_1 to z_0 lies in H_t except the endpoint z_1, the end point z_1 is accessible,[3] see [9] Section 2, Exercise 5, and thus there exists $x \in \mathbb{R}$ such that $z_1 = \lim_{y\to 0} f_t(x + \mathrm{i}y)$. From the facts that $\sup\{|z - W_{t-\delta}| : z \in g_{t-\delta}(K_t \setminus K_{t-\delta})\} = o(1)$ as $\delta \to 0$, see Theorem 4.2, and that $s \mapsto W_s$ is continuous, it follows that $x = W_s$. □

Let $H \subset \mathbb{H}$. Denote by $\partial_+ H$ the set points, which have the property that every neighborhood (in $\mathbb{C}$) of the point intersects both H and $\mathbb{H} \setminus H$. In other words, $\partial_+ H$ is the boundary of H in $\mathbb{H}$.

Remember that $(K_t)_{t\in[0,T)}$ is generated by a curve $\gamma : [0,T) \to \mathbb{C}$, if H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. The next result is a basic tool to verify that a Loewner chain is generated by a curve.

Theorem 6.4. *If $t \mapsto F_t(y)$ converges to some γ uniformly on compact subsets of $[0,T)$ as $y > 0$ tends to 0, then γ is a continuous curve and $(K_t)_{t\in[0,T)}$ is generated by γ. Furthermore for each $t \in [0,T)$, the map $z \mapsto f_t(z)$ extends continuously to $\overline{\mathbb{H}}$.*

Proof. For each fixed $y > 0$, the map $t \mapsto F_t(y)$ is continuous by Lemma 6.1. Hence the uniform convergence of $F_t(y) \to \gamma(t)$ as $y \to 0$ on compact subsets of $[0,T)$ implies that $\gamma : [0,T) \to \mathbb{C}$ is continuous.

It remains to show that for each $t \in [0,T)$, H_t is the unbounded component of $\mathbb{H} \setminus (\gamma[0,t])$. Since $\gamma(s) \in \partial_+ H_s$, it follows that $\gamma[0,t] \subset \bigcup_{s\in[0,t]} \partial_+ H_s$. Hence it is sufficient to show that $\partial_+ H_t \subset \gamma[0,t]$.

Let $z_0 \in \partial_+ H_t$. If $z_0 = W_0$, then clearly $z_0 \in \gamma[0,t]$. Suppose then that $z_0 \neq W_0$ and take any $\varepsilon \in (0, |z_0 - W_0|)$. Let

[3] A boundary point is accessible, if there is a Jordan arc in the domain ending at that point. If we apply a conformal map from the domain onto $\mathbb{D}$, say, then the image of that arc is continuous up to the boundary. Consequently, the accessible point is always a limit of a image of a Jordan arc in $\mathbb{D}$ under a conformal map from $\mathbb{D}$ onto the domain. The radial limit of the conformal map at the same boundary point of $\mathbb{D}$ follows from Corollary 2.17 of [9].

$$t_\varepsilon = \inf\left\{ s \in \mathbb{R}_{\geq 0} \,:\, K_s \cap \overline{B(z_0,\varepsilon)} \neq \emptyset \right\}.$$

Then $0 < t_\varepsilon < t$. Since $z_0 \in \partial_+ H_t$, the set $\overline{\mathbb{H} \cap B(z_0,\varepsilon)} \setminus K_{t_\varepsilon}$ is non-empty. By Lemma 6.5, $|z_0 - \gamma(t_\varepsilon)| = \varepsilon$. Therefore $z_0 = \lim_{\varepsilon\to 0} \gamma(t_\varepsilon)$ and consequently $z_0 \in \overline{\gamma[0,t)} = \gamma[0,t]$. Thus $\partial_+ H_t \subset \gamma[0,t]$ as was claimed. The set H_t is therefore the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ and since ∂H_t is locally connected f_t extends continuously to $\overline{\mathbb{H}}$ as claimed. □

To stress the importance of the previous result, let us formulate the following corollary.

Corollary 6.1. *A Loewner chain $(K_t)_{t\in[0,T)}$ is generated by a curve if and only if for each $T' \in [0,T)$, there exists a function $\lambda : (0,1] \to \mathbb{R}_{\geq 0}$ such that $\lim_{y\to 0} \lambda(y) = 0$ and*

$$|F_t(y_1) - F_t(y_2)| \leq \lambda(y) \tag{6.24}$$

for all $y \in (0,1]$, $y_1, y_2 \in (0,y]$ and $t \in [0,T']$.

6.2.2 *Proof of Theorem 5.2*

6.2.2.1 Auxiliary results on conformal maps

The next result is a version of Koebe distortion theorem in $\mathbb{H}$. The proof, which is straightforward, is given in appendices.

Lemma 6.6 (Koebe distortion in $\mathbb{H}$). *There exists a constant C such that for any $y > 0$, $s \in [\frac{1}{2},2]$, $x \in \mathbb{R}$ and any conformal map $f : \mathbb{H} \to \mathbb{C}$,*

$$C^{-1}|f'(\mathrm{i}y)| \leq |f'(\mathrm{i}sy)| \leq C|f'(\mathrm{i}y)| \tag{6.25}$$

$$C^{-1}(1+x^2)^{-3}|f'(\mathrm{i}y)| \leq |f'(y(x+\mathrm{i}))| \leq C(1+x^2)^3|f'(\mathrm{i}y)|. \tag{6.26}$$

The next result is based on the Loewner equation and thus the proof is given here.

Lemma 6.7. *There exists a constant C such that for any solution f_t of the Loewner equation for the inverse Loewner map and for any $x+\mathrm{i}y \in \mathbb{H}$, $t \in \mathbb{R}_{\geq 0}$ and $s \in [0,y^2]$*

$$C^{-1}\,|f_t'(x+\mathrm{i}y)| \leq |f_{t+s}'(x+\mathrm{i}y)| \leq C\,|f_t'(x+\mathrm{i}y)| \tag{6.27}$$

$$|f_{t+s}(x+\mathrm{i}y) - f_t(x+\mathrm{i}y)| \leq C\,y\,|f_t'(x+\mathrm{i}y)|. \tag{6.28}$$

Proof. By differentiating the Loewner equation and using the triangle inequality and the inequality $|x+\mathrm{i}y - W_t| \geq y$, it follows that

$$|\partial_t f_t'(x+\mathrm{i}y)| \leq \frac{2|f_t''(x+\mathrm{i}y)|}{y} + \frac{2|f_t'(x+\mathrm{i}y)|}{y^2}.$$

To estimate $|f_t''(z)|$, for fixed $z = x + \mathrm{i}y \in \mathbb{H}$, let

$$\phi(\zeta) = x + \mathrm{i}y \frac{1-\zeta}{1+\zeta}.$$

Then ϕ is a Möbius map from $\mathbb{D}$ onto $\mathbb{H}$ and it has expansion

$$\phi(\zeta) = x + \mathrm{i}y \left(1 + 2 \sum_{n=1}^{\infty} (-1)^n \zeta^n \right).$$

Thus $\phi(0) = z$, $\phi'(0) = -2\mathrm{i}y$ and $\phi''(0) = 4\mathrm{i}y$.

The function $(f_t \circ \phi(\zeta) - f_t(z))/(f_t'(z)\phi'(0))$ has expansion

$$\zeta + \frac{f_t''(z)(\phi'(0))^2 + f_t'(z)\phi''(0)}{2 f_t'(z)\phi'(0)} \zeta^2 + \dots$$

around $\zeta = 0$. Using the equation (3.9), it follows that

$$|f_t''(z)||\phi'(0)|^2 \le |f_t'(z)|(|\phi''(0)| + 4|\phi'(0)|)$$

and thus

$$|f_t''(z)| \le 6 \frac{|f_t'(z)|}{y}.$$

Combining this with the above estimate gives

$$|\partial_t f_t'(x+\mathrm{i}y)| \le \frac{14|f_t'(x+\mathrm{i}y)|}{y^2}.$$

Thus $|\partial_t \log f_t'(x+\mathrm{i}y)| \le \frac{14}{y^2}$ and hence

$$-\frac{14}{y^2} \le \partial_t \log |f_t'(x+\mathrm{i}y)| \le \frac{14}{y^2}$$

where we used the inequality $-|z| \le \mathrm{Re}\, z \le |z|$.

By integrating this inequality with respect to t, we get the first claim easily. The second claim is derived from the first one by plugging it in to the Loewner equation, which is then integrated with respect to t. This gives an upper bound which is proportional to $|f_t'(x+\mathrm{i}y)| \frac{s}{y} \le |f_t'(x+\mathrm{i}y)| y$. □

6.2.2.2 The proof

Definition 6.3. An increasing, continuous function $\psi : [0,\infty) \to (0,\infty)$ is said to be *subpower function* if

$$\lim_{x\to\infty} \frac{\log \psi(x)}{\log x} = 0.$$

or equivalently if for all $\mu > 0$ $\lim_{x\to\infty} x^{-\mu}\psi(x) = 0$.

Remark 6.5. One way to write this is $\psi(x) = \exp(o(\log x))$. If ψ_1 and ψ_2 are subpower functions also $\psi_1\psi_2$, $\psi_1+\psi_2$ and $\psi(x) = \psi_1(x^p)$, $p > 0$, are subpower functions.

Proof (Proof of Theorem 5.2). By Theorem 6.4, it is enough to prove that the functions $t \mapsto f_t(W_t + \mathrm{i}y)$ converges uniformly as y tends to 0.

Our goal is to prove this based on the following bounds: As we saw above for each $\kappa \neq 8$, there exist a constant $\theta > 0$ and a random variable C which is almost surely finite such that

$$|\tilde{f}_t'(i2^{-n})| \leq C\,2^{n(1-\theta)} \tag{6.29}$$

for all $t \in \mathscr{D}_{2n}$ and for any $n \in \mathbb{N}$. Remember also that since $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ is a Brownian motion, there is an almost surely finite random variable $\tilde{C}$ such that

$$|W_{t+s} - W_t| \leq \tilde{C}\sqrt{s\log(1/s)} \tag{6.30}$$

for any $t,s \in [0,1]$. Fix a realization of the driving process and the Loewner chain such that the bounds (6.29) and (6.30) hold for some finite C and $\tilde{C}$.

Let $t \in [0,1], y \in (0,1)$. Take $n \in \mathbb{N}$ and $t_0 \in \mathscr{D}_{2n}$ such that

$$2^{-n} \leq y < 2^{-n+1}, \qquad t_0 \leq t < t_0 + 2^{-2n},$$

that is, $n = \lceil \log_2(1/y)\rceil$ and $t_0 = \lfloor t2^{2n}\rfloor 2^{-2n}$. By (6.29), (6.30) and Lemmas 6.6 and 6.7, it follows that

$$\begin{aligned}
|\tilde{f}_t'(iy)| &= |f_t'(W_t+iy)| \leq c|f_{t_0}'(W_t+iy)| \\
&\leq c|f_{t_0}'(W_t+iy_0)| \leq c\left(1+\frac{|W_t-W_{t_0}|^2}{y_0^2}\right)^3 |f_{t_0}'(W_{t_0}+iy)| \\
&\leq cn^r 2^{n(1-\theta)} \leq y^{\theta-1}\psi(1/y)
\end{aligned} \tag{6.31}$$

for some subpower function ψ. Here c is a generic constant that might change from line to line .

Let's integrate the bound (6.31). For any $0 < y_1 < y_2 \leq y < 1$, by the triangle inequality and a change of integration variable

$$|\tilde{f}_t(\mathrm{i}y_2) - \tilde{f}_t(\mathrm{i}y_1)| \leq \int_{y_1}^{y_2} |\tilde{f}_t'(\mathrm{i}u)|\,\mathrm{d}u \leq \int_0^y u^{\theta-1}\psi(1/u)\,\mathrm{d}u = y^\theta\tilde{\psi}(1/y)$$

where

$$\tilde{\psi}(x) = \int_0^1 u^{\theta-1}\psi(x/u)\,\mathrm{d}u.$$

It is not difficult to check that $\tilde{\psi}$ is a subpower function. Hence

$$\gamma(t) = \lim_{y\searrow 0}\tilde{f}_t(\mathrm{i}y)$$

exists and satisfies

$$|\gamma(t) - \tilde{f}_t(\mathrm{i}y)| \leq y^\theta \tilde{\psi}(1/y). \tag{6.32}$$

Now by Theorem 6.4, γ continuous and generates $(K_t)_{t\in\mathbb{R}_{\geq 0}}$. □

In fact, it is possible to prove the following result. The proof is given in appendices.

Proposition 6.5. *For each $\kappa \neq 8$, there exist a constant $\alpha_0 > 0$ such that $t \mapsto \gamma(t)$ is Hölder continuous for any exponent $\alpha < \alpha_0$*

Remark 6.6. We can choose $\alpha_0 = \theta_0/2$ where θ_0 is as in the remark after Proposition 5.11.

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