Chapter 5
Schramm–Loewner evolution

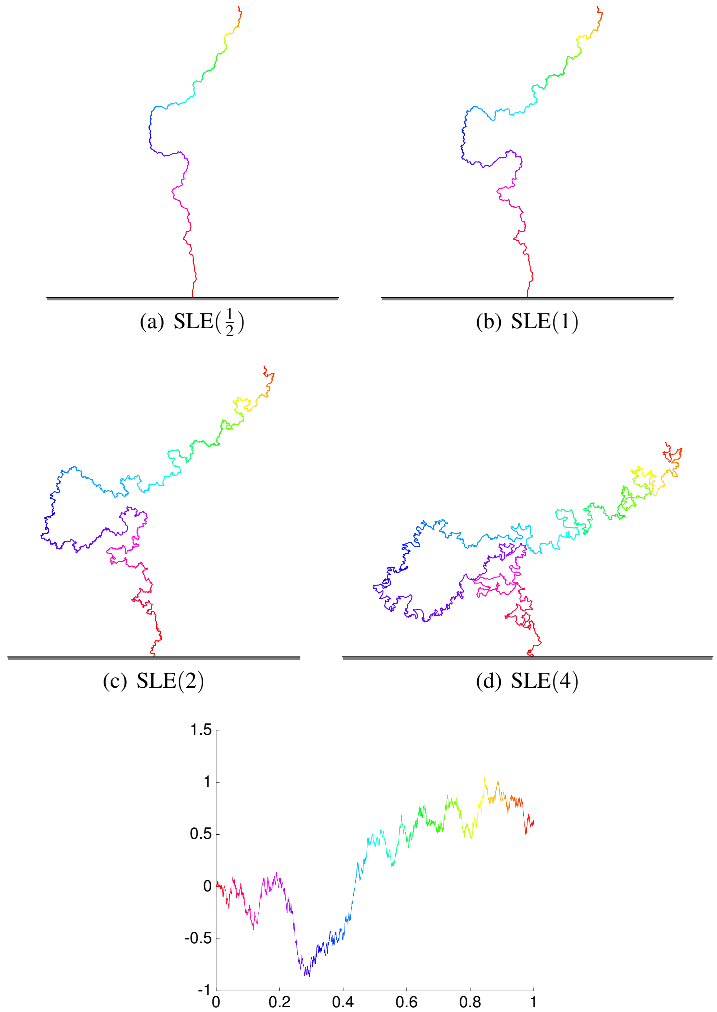
5.1 Schramm–Loewner evolution and its elementary properties

5.1.1 Introduction

The *Schramm–Loewner evolutions*, which first were called *stochastic Loewner evolutions* (for instance, in Schramm's original article [11]), were invented by Oded Schramm[1]. His groundbreaking innovation in his paper released in 1999 was that random curves can be described using the Loewner equation with a random driving term. This enabled him to define Schramm–Loewner evolutions, which in general are Loewner chains driven by stochastic processes which have a Markov type property and which locally resemble Brownian motions.

The motivation of studying SLE comes from statistical physics and predictions of theoretical physics. Given a lattice model of statistical physics with a tempereture-like parameter, for instance, the Ising model, it is believed based on so called Renormalization group analysis that there is a critical value of this parameter so that which separates two regimes: in the large system limit, inside each of the regimes the system looks macroscopically the same for all parameter values. We can say that the system renormalizes either to the zero temperature system or the infinite temperature system. In between the regimes there is a *critical parameter*. In the large system limit, it doesn't renormalize to either of the above mentioned fixed points, but it will be third fixed point of the system. Because of the renormalization phenomena (the fact that larger systems can be seen as slightly smaller systems with renormalized parameters through coarse-graining) one expects that the fixed points are *scale invariant*.

[1] Oded Schramm (1961–2008) was an Israeli-American mathematician, who was a highly influential researcher in the fields of complex analysis and probability theory and is best known for inventing SLE and deriving many of its properties (with his co-authors) as well as many other insightful results around random processes related to statistical physics. He died tragically in a climbing accident.



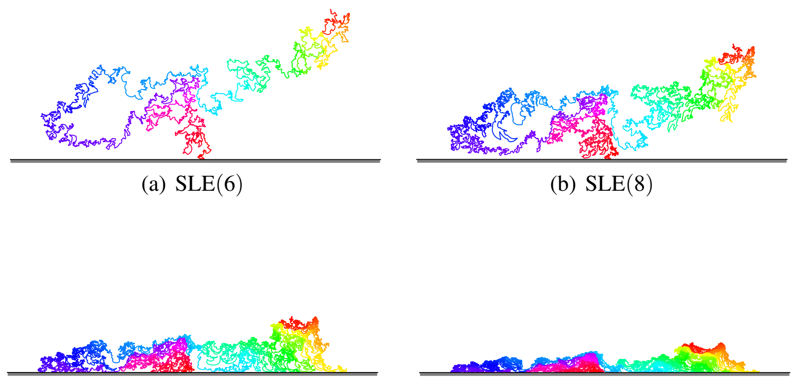
(a) SLE($\frac{1}{2}$) (b) SLE(1)

(c) SLE(2) (d) SLE(4)

(e) The standard Brownian motion sample driving the SLE's.

Fig. 5.1 Instances of SLE(κ), for $\kappa = \frac{1}{2}, 1, 2, 4$, which is a stochastic Loewner chain driven by continuous driving process (i.e., stochastic driving term), which is a standard Brownian motion (drawn in Figure (e)) multiplied by a factor $\sqrt{\kappa}$.

The predictions of theoretical physics based on the Renormalization group analysis also yield that the critical systems should be *conformally invariant* which is a stronger symmetry than scale invariance. In physics, the continuum theory with conformal symmetry is called *Conformal field theory* (CFT). Based on so called *Schramm's principle*, which we will go through in Section 5.1.2, a random curve with a Markovian property and conformal symmetry is a Schramm–Loewner evo-



(a) SLE(6) (b) SLE(8)

(c) SLE(16) (d) SLE(32)

Fig. 5.2 Instances of SLE(κ), for $\kappa = 6, 8, 16, 32$, driven by the instance of a standard Brownian motion drawn in Figure 5.1(e) multiplied by a factor $\sqrt{\kappa}$.

lution (SLE), that is, it is a stochastic Loewner chain with a Brownian motion as its (stochastic) driving term.

Scale invariant random curves can be either smooth or fractal, but if the large system limit of a random curve is probabilistically non-trivial, then the curve has to be a fractal. If there are fluctuations on some given scale, similar fluctuations can be seen in all the other scales, too, by scale invariance. This fractal feature of SLE's is well seen in Figures 5.1 and 5.2.

5.1.2 Schramm's principle

In this section, we present *Schramm's principle* which is a calculation that characterizes a family of random Loewner chains that have connection to statistical physics.

We present this principle on a heuristical level, but with some additional definitions and assumptions this could be made a theorem that states the following.

Schramm's principle. *Schramm–Loewner evolutions are the only random curves satisfying conformal invariance and the domain Markov property.*

We expect those two properties to be satisfied by scaling limits of random interfaces of statistical physics models at criticality as discussed above in this chapter.

Assume that we are given a collection of probability measures $(\mu^{(U,a,b)})$ indexed by the set of all triplets (U,a,b) where U is any simply connected domain and $a \neq b$ are any two boundary points of U. Assume that $\mu^{(U,a,b)}$ is the law of a random curve $\gamma : [0,\infty) \to \mathbb{C}$ (the parametrization is arbitrary) such that $\gamma([0,\infty)) \subset \overline{U}$ and $\gamma(0) = a$, $\gamma(\infty) = b$. We assume that the family $(\mu^{(U,a,b)})$ satisfies the following properties:

1. Let ϕ_* denote the *pushforward* defined by $\phi_* \mathsf{P} = \mathsf{P} \circ \phi^{-1}$. The family $(\mu^{(U,a,b)})$ satisfies **conformal invariance** (CI): for all (U,a,b)

$$\phi_* \mu^{(U,a,b)} = \mu^{(\phi(U),\phi(a),\phi(b))}$$

2. Let $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ be the filtration generated by $(\gamma(t))_{t\in\mathbb{R}_{\geq 0}}$. The family $(\mu^{(U,a,b)})$ satisfies **domain Markov property** (DMP): for all (U,a,b), for every (random[2]) $t \in \mathbb{R}_{\geq 0}$ and for any measurable set B in the space of curves (in what ever way that space is defined...)

$$\mu^{(U,a,b)}(\gamma|_{[t,\infty)} \in B \,|\, \mathscr{F}_t) = \mu^{(U\setminus\gamma([0,t]),\gamma(t),b)}(\gamma \in B).$$

3. We also assume that we can describe the curve γ by the Loewner equation in the sense that there is a $\mu^{(\mathbb{H},0,\infty)}$-almost sure event on which the random curve satisfies Theorem 4.2.

In Schramm's principle, we'll investigate the consequences of these assumptions.

The first observation is that we need to describe only one of the measures in the family. Then CI fixes the rest of them. Let us choose to work with $\mu^{(\mathbb{H},0,\infty)}$. By Theorem 4.2 for each realization of γ there is a driving term $(W_t(\gamma))_{t\in\mathbb{R}_{\geq 0}}$ such that the corresponding conformal maps g_t satisfy the Loewner equation. Here we also make a reparametrization with the half-plane capacity.[3] Let's call the stochastic driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ as *driving process* of the random curve γ.

Fix some $t \in \mathbb{R}_{\geq 0}$. Define $\hat{\gamma}(s) = g_t(\gamma(t+s)) - W_t$ for all $s \in \mathbb{R}_{\geq 0}$. By CI and the DMP, $\hat{\gamma}$ is distributed as γ and independent of the realization of $\gamma|_{[0,t]}$. The conformal map associated to the hull $\hat{\gamma}([0,s])$ is

$$\hat{g}_s(z) = \tilde{g}_{t,s}(z+W_t) - W_t = g_{t+s} \circ g_t^{-1}(z+W_t) - W_t.$$

Now by differentiating this with respect to s

$$\begin{aligned}\partial_s \hat{g}_s(z) &= (\partial_s g_{t+s})(g_t^{-1}(z+W_t)) \\ &= \frac{2}{g_{t+s}(g_t^{-1}(z+W_t)) - W_{t+s}} = \frac{2}{\hat{g}_s(z) - (W_{t+s} - W_t)}\end{aligned}$$

Hence the driving process of $\hat{\gamma}$ is

$$\hat{W}_s = W_{t+s} - W_t$$

Since $\hat{\gamma}$ is distributed as γ and is independent of $\mathscr{F}_t$, which is the σ-algebra generated by $\gamma(s)$, $s \in [0,t]$, $(\hat{W}_s)_{s\in\mathbb{R}_{\geq 0}}$ is independent of $\mathscr{F}_t$ and it is distributed as $(W_s)_{s\in\mathbb{R}_{\geq 0}}$. Hence the continuous stochastic process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ has *independent and stationary increments*. Theorem 2.3 shows that $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ is a Brownian motion with drift.

Now the driving process of a random curve γ distributed according to $\mu^{(\mathbb{H},0,\infty)}$ is

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[2] Technically here we should restrict to so called stopping times.

[3] By an argument which we leave as an exercise, when CI and DMP are satisfied, the half-plane capacity of the hull $\gamma[0,t]$ will tend to infinity as t tends to infinity. Therefore the reparametrized curve will be parametrized by the set $\mathbb{R}_{\geq 0}$.

$$W_t = \sqrt{\kappa} B_t + \alpha t$$

for some $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. We will show that $\alpha = 0$. We apply once more CI and note that $\mu^{(\mathbb{H},0,\infty)}$ is invariant under any scaling $z \mapsto \lambda z$, $\lambda > 0$, that is, $\gamma^{(\lambda)}$ defined by $\gamma^{(\lambda)}(t) = \lambda \gamma(t/\lambda^2)$ is distributed as γ. Note that the correct parametrization of $\gamma^{(\lambda)}$ follows from the scaling property of the half-plane capacity. By a similar calculation as above, it follows that the driving process of $\gamma^{(\lambda)}$ is

$$W_t^{(\lambda)} := \lambda W_{t/\lambda^2}.$$

Since $(W_t^{(\lambda)})_{t \in \mathbb{R}_{\geq 0}}$ is distributed as $(W_t)_{t \in \mathbb{R}_{\geq 0}}$, the driving process satisfies the Brownian scaling and hence $\alpha = 0$ and

$$W_t = \sqrt{\kappa} B_t.$$

To conclude this section, we have shown that the only families $(\mu^{(U,a,b)})$ satisfying CI and the DMP are those where the measure $\mu^{(\mathbb{H},0,\infty)}$ is the law of a random curve with a Loewner driving process equal to a constant multiple of a one-dimensional Brownian motion.

5.1.3 Definition of SLE as a stochastic Loewner chain

We wish to define a random Loewner chain. We start by a short comment on the measurability of such a construction.

Based on similar approach with the theory of ODE as Lemma 4.9 we can show the following result.

Lemma 5.1. *The mapping from $(W_t)_{t \in \mathbb{R}_{\geq 0}}$ to $(g_t)_{t \in \mathbb{R}_{\geq 0}}$ is continuous in the following sense: if $(W_t)_{t \in \mathbb{R}_{\geq 0}}$ and $(\tilde{W}_t)_{t \in \mathbb{R}_{\geq 0}}$ are continuous function and $(g_t)_{t \in \mathbb{R}_{\geq 0}}$ and $(\tilde{g}_t)_{t \in \mathbb{R}_{\geq 0}}$ are the corresponding Loewner chains with hulls $(K_t)_{t \in \mathbb{R}_{\geq 0}}$ and $(\tilde{K}_t)_{t \in \mathbb{R}_{\geq 0}}$, respectively, and $J \subset \mathbb{H} \setminus K_T$ is compact, then for some constants $C > 0$ and $\varepsilon > 0$ independent of $(\tilde{W}_t^{(n)})_{t \in \mathbb{R}_{\geq 0}}$, it holds that $\|g_T - \tilde{g}_T\|_{\infty,J} \leq C \|W - \tilde{W}\|_{\infty,[0,T]}$ whenever $\|W - \tilde{W}\|_{\infty,[0,T]} < \varepsilon$.*

By Lemma 5.1, the mapping from the continuous functions $(W_t)_{t \in \mathbb{R}_{\geq 0}}$ to the corresponding Loewner chains $(g_t)_{t \in \mathbb{R}_{\geq 0}}$ (the solution of (4.14)) is continuous if we use the following topologies in these spaces. The topology of the driving functions is given by the locally uniform convergence, that is, a sequence converges if it converges uniformly on every compact subinterval of $\mathbb{R}_{\geq 0}$. The topology of the Loewner chains is given by a form of Carathéodory convergence. More specifically, a sequence of Loewner chains[4] $(g_n(t,\cdot), K_n(t))_{t \in \mathbb{R}_{\geq 0}}$ converges

[4] We use here a variant of the notation so that $W_t, K_t, g_t(z)$ etc. are replaced by $W(t), K(t), g(t,z)$.

to $(g(t,\cdot),K(t))_{t\in\mathbb{R}_{\geq 0}}$, if for any $T>0$ and any compact $J\subset\mathbb{H}\setminus K_T$, the sequence of functions $(t,z)\mapsto g_n(t,z)$ converges uniformly to $(t,z)\mapsto g(t,z)$ on $[0,T]\times J$.

In particular the map of the previous paragraph is measurable and hence if we have a probability space $(\Omega,\mathscr{F},\mathsf{P})$ with a continuous stochastic process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$, we can define a Loewner chain valued random variable $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ corresponding to the stochastic driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$. We call them a *driving process* and a *stochastic Loewner chain*.

Next we define SLE as a stochastic Loewner chain. This is somewhat unsatisfactory since ultimately we aim to define it as a random curve — the point of view which we took earlier in this chapter.

Definition 5.1 (Chordal SLE in $\mathbb{H}$). Let $\kappa\geq 0$. A *chordal Schramm–Loewner evolution* SLE(κ) is a stochastic Loewner chain with a driving process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ equal to a Brownian motion with variance parameter κ, that is, $W_t=\sqrt{\kappa}B_t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

Remark 5.1. We call this kind of SLEs *chordal* because we *expect* that they will be random curves that connect two boundary points, namely, 0 and ∞. A *radial* SLE would be a random curve connecting a boundary point to an interior point.

Example 5.1 (SLE(0) is trivial). If W_t is identically zero, then g_t is the solution of ODE $\partial_t g_t(z)=2/g_t(z)$, $g_0(z)=z$, which can be integrated to give $g_t(z)=\sqrt{z^2+4t}$. Therefore SLE(0) is the vertical line segment $t\mapsto 2\sqrt{t}$. See also Example 4.3. To exclude this trivial example, we *make the assumption that* $\kappa>0$.

5.1.4 Elementary properties of SLE

The next theorem captures the elementary consequences of the definition of SLE. For the review of stopping times, Markov properties etc. consult Chapter 2 and references therein.

Theorem 5.1. *Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be SLE(κ), $\kappa>0$, and $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ the corresponding driving process which is a Brownian motion with respect to a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$. SLE$(\kappa)$ satisfies the following properties.*

1. Scale invariance*: For any $\lambda>0$, $(\lambda K_{t/\lambda^2})_{t\in\mathbb{R}_{\geq 0}}\overset{d}{=}(K_t)_{t\in\mathbb{R}_{\geq 0}}$.*
2. Conformal Markov property*: For any $s\in\mathbb{R}_{\geq 0}$, the family of hulls*
$$(\hat{K}_{s,t})_{t\in\mathbb{R}_{\geq 0}}=\left(\overline{g_s(K_{s+t}\setminus K_s)-W_s}\right)_{t\in\mathbb{R}_{\geq 0}}$$
is independent of $\mathscr{F}_s$ and $(\hat{K}_{s,t})_{t\in\mathbb{R}_{\geq 0}}\overset{d}{=}(K_t)_{t\in\mathbb{R}_{\geq 0}}$.
3. Strong conformal Markov property*: For any almost surely finite stopping time τ with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_{\geq 0}}$, the family of hulls*
$$(\hat{K}_{\tau,t})_{t\in\mathbb{R}_{\geq 0}}=\left(\overline{g_\tau(K_{\tau+t}\setminus K_\tau)-W_\tau}\right)_{t\in\mathbb{R}_{\geq 0}}$$

is independent of $\mathscr{F}_\tau$ *and* $(\hat{K}_{\tau,t})_{t\in\mathbb{R}_{\geq 0}} \stackrel{d}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$.

Proof. In the cases 1–3, the hulls and the corresponding conformal maps are

$$\begin{array}{rrr} \lambda K_{t/\lambda^2}, & \overline{g_s(K_{s+t}\setminus K_s)-W_s}, & \overline{g_\tau(K_{\tau+t}\setminus K_\tau)-W_\tau} \\ \lambda g_{t/\lambda^2}(z/\lambda), & \hat{g}_{s,t}(z), & \hat{g}_{\tau,t}(z), \end{array}$$

respectively, where

$$\hat{g}_{s,t}(z) = g_{s+t}\circ g_s^{-1}(z+W_s)-W_s.$$

By differentiating these functions with respect to t, we find that the Loewner chains satisfy the Loewner equation with the driving processes

$$\lambda W_{t/\lambda^2}, \qquad W_{s+t}-W_s, \qquad W_{\tau+t}-W_\tau,$$

respectively. The claims now follow from the scaling property, the Markov property and the strong Markov property of Brownian motion. □

We leave as an exercise to verify that SLE is also *symmetric* under the reflection with respect to the y-axis, that is, $(m(K_t))_{t\in\mathbb{R}_{\geq 0}} \stackrel{d}{=} (K_t)_{t\in\mathbb{R}_{\geq 0}}$, where $m(z) = -\bar{z}$.

5.1.5 SLE in a general simply-connected domain

At the moment, we have defined only $\mu^{(\mathbb{H},0,\infty)}$. Guided by the Schramm's principle, we now extend the definition to include $\mu^{(U,a,b)}$ for a general simply connected domain with two distighuished boundary points. It is natural to use the conformal invariance requirement for doing this and define SLE(κ) in other domains by the conformal image of a SLE(κ) in $\mathbb{H}$. This definition relies on the fact that SLE(κ) in $\mathbb{H}$ started from $W_0 = 0$ is scale invariant, see Theorem 5.1.

Definition 5.2 (Chordal SLE in a general simply connected domain). Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be a chordal SLE(κ) and let U be a simply connected domain and a and b two boundary points of U with $a \neq b$. We define *(chordal) SLE*(κ) *in a domain U going from a to b* to be the image of $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ under any conformal onto map $\phi : \mathbb{H}\to U$ with $\phi(0) = a$ and $\phi(\infty) = b$.

Remark 5.2. This definition is unique only up to a linear time change, because all the conformal onto maps from $\mathbb{H}$ to U with the above properties are of the form $z \mapsto \phi(\lambda z)$ where $\lambda > 0$ is a constant. By the scaling property of SLE, the choice of this conformal map only affects the time parametrization of the hulls in U.

Remark 5.3. If the boundary of U is not locally connected and ϕ doesn't extend continuously to the boundary, a and b has to be understood as "generalized boundary points", more specifically as prime ends.

Naturally we make here the exception that SLE(κ) in $\mathbb{H}$ from x to ∞ will always have parametrization with the half-plane capacity and therefore it is defined as the solution of the Loewner equation with the driving process $W_t = x+\sqrt{\kappa}B_t$.

5.2 Advanced properties of SLE

In this section, we review some properties of SLE. The proofs of most these facts will be given in the later sections or chapters.

5.2.1 SLE is generated by a curve

At this point SLE(κ) is a stochastic Loewner chain. It turns out that it can be defined as a random curve in the sense of the theorem below.

Definition 5.3. A growing family of hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ is *generated by a curve* γ if $\mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all $t\in\mathbb{R}_{\geq 0}$.

For any Loewner chain g_t, $t\in\mathbb{R}_{\geq 0}$, we try to define the generating curve γ as $\gamma(t)=\lim_{\varepsilon\searrow 0} f_t(W(t)+i\varepsilon)$ where $f_t=g_t^{-1}$. The function γ, if it exists, is called the *trace* of the Loewner chain.

Theorem 5.2. *For each κ, the trace γ exists and is a random curve such that the hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ of SLE(κ) are generated by γ almost surely.*[5]

We will *make the assumption that the previous result holds for SLE(κ)*. We will use that assumption without mentioning it. The result will be proven in Chapter 6 using estimates established for $f_t'(z+\mathrm{i}W_t)$ in this chapter.

5.2.2 Phases of SLE

The next theorem summarizes the important facts on the random curve γ. We will prove those statements at least partly in this section and we are going to do it in several stages.

Theorem 5.3. *Let the random curve $\gamma:[0,\infty)\to\overline{\mathbb{H}}$ be SLE(κ) (in the sense of Theorem 5.2). Then*

- *For all $0<\kappa\leq 4$, γ is simple and $\gamma(0,\infty)\cap\mathbb{R}=\emptyset$.*
- *For all $4<\kappa<8$, γ is not simple:*

$$\text{for any } 0\leq t_1<t_2 \text{ there exists } t_1<s_1<s_2<t_2 \text{ such that } \gamma(s_1)=\gamma(s_2). \quad (5.1)$$

  However, γ is not space-filling: for any $z\in\mathbb{H}$, $\mathsf{P}[\mathrm{dist}(z,\gamma[0,\infty))>0]=\mathsf{P}[z\notin\gamma[0,\infty)]=1$.
- *For all $\kappa\geq 8$, γ is not simple, it satisfies 5.1, but γ is space-filling: $\mathsf{P}[z\in\gamma[0,\infty)]=1$.*

[5] We will prove the theorem only for $\kappa\neq 8$. The case $\kappa=8$ is a consequence of the results of [9].

Moreover, γ is transient in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$.

Remark 5.4. A way to formulate transience is that under the law $\mu^{(U,a,b)}$ (as in Schramm's principle) the random curve γ tends to b as the time parameter tends to its terminal value.

5.2.3 Dimension of SLE

Remember that the *Hausdorff dimension* $d = \dim_{\mathscr{H}}(\Gamma)$ of a point set $\Gamma \subset \mathbb{C}$ is such that the s-dimensional Hausdorff measure[6] $\mathscr{H}^s[\Gamma]$ is infinite for all $s < d$ and zero for all $s > d$, or more definitely, $\dim_{\mathscr{H}}(\Gamma) = \inf\{s \geq 0 : \mathscr{H}^s[\Gamma] < \infty\}$.

The following result shows that SLE(κ)'s are random fractals, which we anticipated based on the statistical scale invariance. It also gives a very nice interpretation for the parameter κ.

Theorem 5.4. *For $\kappa > 0$, let $\Gamma = \gamma[0,\infty)$, where γ is the trace of SLE(κ). Then* $\dim_{\mathscr{H}}(\Gamma) = 2 \wedge (1 + \frac{\kappa}{8})$.

Notice that since $\Gamma \subset \mathbb{C}$, $\dim_{\mathscr{H}}(\Gamma) \leq 2$, and that for SLE($\kappa$), it holds that $\dim_{\mathscr{H}}(\Gamma) = 2$ if and only if γ is space-filling.

5.3 Proofs of some of the advanced properties

5.3.1 SLE and Bessel processes

We start the investigation to prove Theorem 5.3 by continuing our review of topics in stochastic analysis.

5.3.1.1 SLE as a complex Bessel process

Fix $z \in \overline{\mathbb{H}}$ with $z \neq 0$ for a moment. Let g_t be a chordal SLE(κ) with a driving process $W_t = -\sqrt{\kappa} B_t$, where the minus sign is for convenience. Define the processes

$$\hat{Z}_t = g_t(z) - W_t, \qquad Z_t = \hat{Z}_t / \sqrt{\kappa}.$$

[6] The s-dimensional Hausdorff measure $\mathscr{H}^s$ is defined by

$$\mathscr{H}^s[\Gamma] = \lim_{\delta \to 0} \inf \left\{ \sum_{k=1}^{\infty} (\mathrm{diam}(V_k))^s : \Gamma \subset \bigcup_{k=1}^{\infty} V_k \text{ and } \mathrm{diam}(V_k) < \delta \right\}$$

where the infimum is over all countable covers V_k, $k = 1, 2, \ldots, n$, of Γ satisfying $\mathrm{diam}(V_k) < \delta$ for all k.

By the Loewner equation, these processes have the Itô differentials[7]

$$\mathrm{d}\hat{Z}_t = \frac{2}{\hat{Z}_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \mathrm{d}Z_t = \frac{2/\kappa}{Z_t}\mathrm{d}t + \mathrm{d}B_t.$$

Therefore $(Z_t)_{t\in[0,\tau(z))}$, where $\tau(z)$ is as in the section 4.2.2, could be called as a *$\delta(\kappa)$-dimensional complex Bessel process sent from $z/\sqrt{\kappa}$* where

$$\delta(\kappa) = 1 + \frac{4}{\kappa} \in (1,\infty).$$

The standard use of the parameter δ in the context of Bessel processes is presented below in the stochastic differential equation (5.2).

5.3.1.2 Some properties of Bessel processes

In the next proposition we list some properties of (real-valued) Bessel processes.

Proposition 5.1. *Let $\delta \in \mathbb{R}$ and let $(X_t)_{t\in[0,T)}$ be a* δ-dimensional Bessel process sent from $x > 0$*, that is, $(X_t)_{t\in[0,T)}$ is the unique solution*[8] *of*

$$\mathrm{d}X_t = \frac{\delta - 1}{2X_t}\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x \tag{5.2}$$

and $T \in (0,+\infty]$ is the maximal time such that the solution exists and is positive for any $t \in [0,T)$. Then

1. $\mathsf{P}[T < \infty] = 1$ *if and only if* $\delta < 2$*,*
2. $\mathsf{P}[T = \infty] = 1$ *if and only if* $\delta \geq 2$*,*
3. $\mathsf{P}\left[\inf_{0\leq t<T} X_t > 0\right] = 1$ *if and only if* $\delta > 2$*,*
4. $\mathsf{P}[\lim_{t\to T} X_t = 0] = 1$ *when* $\delta < 2$*.*

Remark 5.5. As we saw in Example 2.3, the Euclidian norm of a d-dimensional Brownian motion is a d-dimensional Bessel process.

By this proposition when $\delta \geq 2$, the Brownian motion of dimension δ won't hit the origin. In the case $\delta = 2$, the Brownian motion will get arbitrarily close to zero, though.

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[7] It is convenient to use complex valued Itô processes. It is understood that an equality of the form $\mathrm{d}Z(t) = \xi(t)\mathrm{d}t + \sum_{k=1}^{n}\zeta_k(t)\mathrm{d}B_k(t)$, where $(B_k(t))_{t\in\mathbb{R}_{\geq 0}}$ are standard one-dimensional Brownian motions, means that the real and imaginary part of both of the sides are equal when we consider $\mathrm{d}t$ and $\mathrm{d}B_k(t)$ to be real.

[8] The existence and uniqueness of the solution follows from Theorem 2.10, for instance, using the following trick. For any $n \in \mathbb{N}$, replace the drift term by a smooth continuation of the function that maps $x \mapsto (\delta-1)/(2x)$, $x > 1/n$ and $x \mapsto 0$, $x < 0$. The drift term and the approximating drift term are identical on the interval $[1/n,+\infty)$ and thus the solutions agree until the process exits the interval.

Proof. These claim can be proven using the fact that for δ-dimensional Bessel process $(X_t)_{t\in[0,T)}$, the process

$$M_t = \begin{cases} X_t^{2-\delta} & \text{when } \delta \neq 2 \\ \log X_t & \text{when } \delta = 2 \end{cases} \tag{5.3}$$

is a local martingale for $t < T$. We leave as an exercise to apply Itô's formula to M_t to verify the claim.

Notice also that the Bessel processes are scale invariant so that the satisfy the Brownian scaling $\lambda X_{t/\lambda^2} \overset{\mathrm{d}}{=} X_t$ for all $\lambda > 0$.

For $t < T$, notice that

$$X_t - B_t = x + \frac{\delta - 1}{2} \int_0^t \frac{\mathrm{d}s}{X_s}.$$

Since $X_s > 0$ for all $s \in [0,T)$ it follows that

$$\begin{cases} X_t \geq B_t & \text{when } \delta \geq 1 \\ X_t \leq B_t & \text{when } \delta \leq 1. \end{cases} \tag{5.4}$$

For $a \in (0,x)$, $b \in (x,\infty)$, $c \in (0,x) \cup (x,\infty)$, define $\tau_c = \inf\{[0,T) : X_t = c\}$ and using it $\tau_{a,b} = \tau_a \wedge \tau_b$. Define also $\tau_0 = \lim_{a\to 0} \tau_a$ and $\tau_{0,b} = \lim_{a\to 0} \tau_{a,b}$ which exist since they are decreasing functions of a. By the comparison inequalities (5.4), we can show that for any $a \in [0,x)$ and $b \in (x,\infty)$, the stopping tim $\tau_{a,b}$ is finite almost surely. For instance, when $\delta \geq 1$, $\tau_{a,b}$ is bounded from above by the time that the Brownian motion hits b, which is almost surely finite. The argument is similar for other values of δ.

Since $(M_{t\wedge\tau_{a,b}})_{t\in\mathbb{R}_{\geq 0}}$ is a bounded martingale and $\tau_{a,b}$ is an almost surely finite stopping time, by the optional stopping theorem

$$f(x) = f(a)\mathsf{P}[X_{\tau_{a,b}} = a] + f(b)\mathsf{P}[X_{\tau_{a,b}} = b] \tag{5.5}$$

where $f(x) = x^{2-\delta}$ when $\delta \neq 2$ and $f(x) = -\log(x)$ when $\delta = 2$. Writing (5.5) in the form

$$\mathsf{P}[\tau_a = \tau_{a,b}] = \frac{f(x) - f(b)}{f(a) - f(b)} \tag{5.6}$$

allows us to make the following conclusions

- When $\delta < 2$, $\lim_{a\to 0} f(a) = 0$. Thus $\mathsf{P}[\tau_0 = \tau_{0,b}] = 1 - \frac{f(x)}{f(b)} > 0$. Notice that $\tau_0 = T$. Thus for some $t > 0$, it holds that $\mathsf{P}[T \leq t] > 0$. By scale invariance of the Bessel process, $p = \mathsf{P}[T = \infty]$ is independent of x. It follows that $p = \mathsf{P}[T = \infty | \mathscr{F}_t]$ on the event $T > t$. Taking expected value from both sides it follows that $p = p\mathsf{P}[T > t]$. Since $\mathsf{P}[T \leq t] > 0$ it follows that $p = 0$.
- When $\delta \geq 2$, $\lim_{a\to 0} f(a) = \infty$. Thus $\mathsf{P}[\tau_0 = \tau_{0,b}] = 0$. It is then possible to argue that $\tau_{0,b} = \tau_b$ tends to ∞ as $b \to \infty$, since there is no blow-up by the solution in

finite time by the uniqueness and existence theorem of SDEs. This shows that $T = \tau_0 = \infty$ almost surely.

- When $\delta > 2$, since $\lim_{b\to\infty} \tau_b = \infty$ almost surely and $\lim_{b\to\infty} f(b) = 0$, it holds that $\mathsf{P}[\tau_{2^{-n}} < \infty] = \frac{f(x)}{f(2^{-n})} = x^{2-\delta}\, 2^{-(\delta-2)n}$ for any $n \in \mathbb{Z}_{>0}$ such that $2^{-n} < x$, which is summable over n. Thus by the Borel–Cantelli lemma, $\inf_{t\in\mathbb{R}_{\geq 0}} X_t > 0$ almost surely.
- When $\delta < 2$, then choose $r_n > 1$ such that $\mathsf{P}[\tau_{r_n x} = \tau_{0,r_n x}] = 2^{-n}$. Then by scale invariance and the Markov property of the Bessel process, $\mathsf{P}[X_t \geq n^{-1}$ for some $t > \tau_{n^{-1} r_n^{-1}}] = 2^{-n}$ for all $n \in \mathbb{Z}_{>0}$ such that $n^{-1} r_n^{-1} < x$. Consequently $\limsup_{t\to T} X_t = 0$.

All the claims follow. □

5.3.2 Phase transition from simple to non-simple curve at $\kappa = 4$

We will show in this subsection that γ is simple for $\kappa \in (0,4]$ and non-simple for $\kappa \in (4,+\infty)$.

Remember that $z \in K_t$ if and only if $\tau(z) \leq t$, where $\tau(z)$ is as in Section 4.2.2. Notice also that for $0 < x_1 < x_2$ or for $x_2 < x_1 < 0$, it holds that $W_t < g_t(x_1) < g_2(x_2)$ or $g_2(x_2) < g_t(x_1) < W_t$, respectively, for all $0 < t < \tau(x_1) \wedge \tau(x_2)$ and consequently, it holds that $\tau(x_1) \leq \tau(x_2)$.

Now we use the fact that the process $X_t = (g_t(x) - W_t)/\sqrt{\kappa}$, $x \in \mathbb{R}\setminus\{0\}$, is a Bessel process of dimension $\delta = 1 + \frac{4}{\kappa}$. Proposition 5.1 applied to X_t shows that $\mathsf{P}[\tau(x) < \infty]$ is equal to 1 when $\kappa > 4$ and 0 when $\kappa \in (0,4)$. This together with the above monotonicity property of $\tau(x)$ implies the following result easily.

Proposition 5.2. *For $0 < \kappa \leq 4$, $(\bigcup_{t\in\mathbb{R}_{\geq 0}} K_t) \cap \mathbb{R} = \{0\}$ and for $\kappa > 4$, $\mathbb{R} \subset \bigcup_{t\in\mathbb{R}_{\geq 0}} K_t$ almost surely. Equivalently, almost surely $\tau(x) = \infty$ for all $x \in \mathbb{R}\setminus\{0\}$, when $\kappa \in (0,4]$, and $\tau(x) < \infty$ for all $x \in \mathbb{R}$, when $\kappa > 4$*

Based on this result, let's first show that SLE(κ), $0 < \kappa \leq 4$, is simple, based on this result. Let $s > 0$ and let x_- and x_+ be the two images of 0 under the map $g_s - W_s$. By the previous proposition and by the conformal Markov property, $\hat{\gamma}(t) = g_s(\gamma(s+t)) - W_s$, $t \in \mathbb{R}_{\geq 0}$, intersect the real axis only at 0. In particular it doesn't intersect $[x_-,0) \cup (0,x_+]$. Since $f_s = g_s^{-1}$ is continuous to the boundary, this implies that

$$\gamma[0,s] \cap \gamma[s,\infty) = \{\gamma(s)\} \tag{5.7}$$

almost surely. In fact this holds almost surely for all s (we can show it first for all rational s and then by continuity to all s). If $t_1 < t_2$ are such that $\gamma(t_1) = \gamma(t_2)$, then pick $t_1 < s < t_2$ such that $\gamma(s) \neq \gamma(t_1)$. Then $\gamma(t_1) = \gamma(t_2)$ contradicts with (5.7). Thus γ is simple.

Let's then show that SLE(κ), $\kappa > 4$, is not simple. Let $0 \leq s_1 < u < s_2$. Let $\hat{x}_- \leq 0 \leq \hat{x}_+$ be such that the image of $\gamma[s_1,u]$ under $g_u - W_u$ is $[\hat{x}_-,\hat{x}_+]$. Since $\tau(1)$

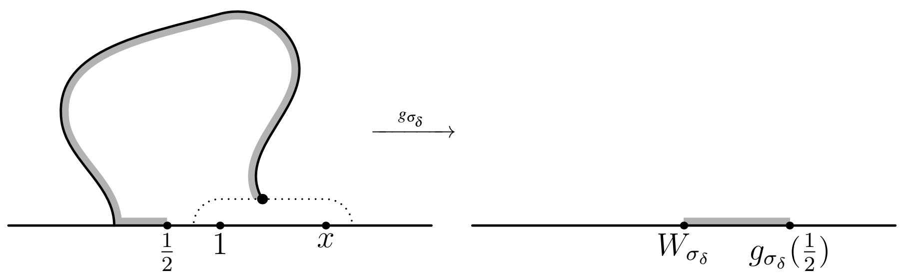

Fig. 5.3 The harmonic function h in the proof of Proposition 5.4 has boundary value 1 in the shaded boundary arc and 0 elsewhere on the boundary. The dotted curve is the set of points at distance δ from the interval $[1,x]$.

is finite almost surely, for fixed $t > 0$, by scaling $\mathsf{P}[\tau(x) \le t] \to 1$ as $x > 0$ tends to 0. Therefore

$$\mathsf{P}[\gamma[0,t] \cap (0,x] \neq \emptyset] = 1$$

for all $t > 0$ and $x > 0$. Hence we can find $u < t_2 < s_2$ such that $g_u(\gamma(t_2)) - W_u \in [\hat{x}_-, 0) \cup (0, \hat{x}_+]$. And hence there exists $s_1 \le t_1 < u$ such that $\gamma(t_1) = \gamma(t_2)$ and we have shown the property (5.1).

We have shown the following result.

Proposition 5.3. *When $\kappa \in (0,4]$, γ is simple almost surely and when $\kappa > 4$, γ is non-simple almost surely in the sense of Theorem 5.3.*

5.3.3 Transience for $\kappa \in (0,4]$

We will show in this subsection that a chordal SLE(κ) curve γ is transient when $\kappa \in (0,4]$, in the sense that $|\gamma(t)| \to \infty$ as $t \to \infty$. In particular, $\overline{\gamma[0,\infty)} = \gamma[0,\infty)$ when γ is transient.

Proposition 5.4. *When $0 < \kappa \le 4$, $\mathsf{P}[\mathrm{dist}(\overline{\gamma[0,\infty)}, [x,x']) > 0] = 1$ for any $0 < x < x'$ or $x < x' < 0$.*

Proof. Suppose first that $\kappa \in (0,4)$. By symmetry and the scale invariance of SLE(κ), it is enough to show that $\mathsf{P}(\mathrm{dist}(\overline{\gamma[0,\infty)}, [1,x]) > 0) = 1$ for all $x > 1$. Let $0 < \delta < 1/4$ and define

$$\sigma_\delta = \inf\{t \in \mathbb{R}_{\ge 0} : \mathrm{dist}(\gamma(t), [1,x]) \le \delta\}. \tag{5.8}$$

Let's consider the event $\sigma_\delta < \infty$. Let R_{σ_δ} be the union of the right-hand side of $\gamma[0,\sigma_\delta]$ and $[0,1/2]$ and $h(z)$ be the bounded harmonic function on $H_{\sigma_\delta} = \mathbb{H} \setminus \gamma[0,\sigma_\delta]$ that has boundary value 1 on R_{σ_δ} and 0 elsewhere. See Figure 5.3. Then

$h(z)$ can be written as the harmonic measure $\mathrm{HM}(z, R_{\sigma_\delta}, H_{\sigma_\delta})$ as in Definition 3.3. By the conformal invariance of harmonic measure, after applying the conformal map g_{σ_δ} we see that for $y \in \mathbb{R}_{>0}$,

$$h(\mathrm{i}y) = \frac{1}{\pi y}(g_{\sigma_\delta}(1/2) - W_{\sigma_\delta}) + \mathscr{O}\left(\frac{1}{y^2}\right)$$

as $y \to \infty$. This can be derived, for instance, by further mapping conformally to the unit disc and sending $g_{\sigma_\delta}(\mathrm{i}y)$ to 0. We leave the details to the reader.

On the other hand, we can write the harmonic measure $h(\mathrm{i}y)$ as the probability that a Brownian motion sent from $\mathrm{i}y$ exits H_{σ_δ} through R_{σ_δ}. On this event the Brownian motion has to intersect the vertical line connecting the interval $[1,x]$ to $\gamma(\sigma_\delta)$. Let $x_0 = \operatorname{Re}\gamma(\sigma_\delta)$. Then the probability that a complex Brownian motion sent from $\mathrm{i}y$ will hit the right-hand side of the segment $[x_0, x_0 + \mathrm{i}\delta]$ before hitting the left-hand side or the real axis is equal to $\frac{\delta}{\pi y} + \mathscr{O}(\frac{1}{y^2})$ as $y \to \infty$. This can be derived similarly as above.

Using the latter harmonic measure as an upper bound for the former one, multiplying by y and taking the limit $y \to \infty$, we conclude that $g_{\sigma_\delta}(1/2) - W_{\sigma_\delta} \le \delta$. Because the infimum of a Bessel process is positive (Proposition 5.1), there exists a positive random variable δ_0 such that $\sigma_\delta = \infty$ for all $\delta \in (0, \delta_0)$. The claim follows for $\kappa \in (0,4)$.

For $\kappa = 4$, it holds that the infimum of a Bessel process with corresponding dimension ($\delta = 2$) is zero. The above argument cannot therefore be used for $\kappa = 4$. One can give an argument similar to that given in Section 5.3.4, see also [10]. □

Proposition 5.5. *For* $0 < \kappa \le 4$, $|\gamma(t)| \to \infty$ *as* $t \to \infty$.

Proof. First of all $|\gamma(t_k)| \to \infty$ along some sequence $t_k \to \infty$, because otherwise γ would be bounded and hence had bounded half-plane capacity.

Let T_1 be the hitting time of $\partial B(0,1)$ by γ and let $x_- < 0 < x+$ be the two images of 0 under the map $g_{T_1} - W_{T_1}$. Then by Proposition 5.4 almost surely distance from the SLE(κ) curve $\hat{\gamma}(t) = g_{T_1}(\gamma(T_1 + t)) - W_{T_1}$, $t \in \mathbb{R}_{\ge 0}$, to $[-n, -1/n] \cup [1/n, n]$ is positive for all $n \in \mathbb{N}$. Hence it will stay at a positive distance from x_- and x_+ and consequently, there exists a random variable $r > 0$ such that $|\gamma(t)| \ge r$ for all $t \ge T_1$. Using this property, scaling and the Borel–Cantelli lemma, we can construct a sequence of random variables $0 < R_1 < R_2 < \ldots$ such that $R_k \to \infty$ almost surely and γ doesn't enter to $B(0, R_{k-1})$ after hitting $\partial B(0, R_k)$. □

5.3.4 Phase transition of distance from a point to γ at $\kappa = 8$

In this subsection, we analyze the distance from z to $\overline{\gamma[0,\tau(z))} = \overline{\gamma[0,\infty)}$.

5.3.4.1 Conformal radius

We first need a conformally covariant[9] version of the distance to the boundary. For any simply connected domain $U \subset \mathbb{C}$ (with $U \neq \mathbb{C}$) and for any $z_0 \in U$, let ψ be the unique conformal map from U onto $\mathbb{D}$ such that $\psi(z_0) = 0$ and $\psi'(z_0) > 0$. Then the *conformal radius of U from z_0* is defined as

$$\rho(z_0, U) = \frac{1}{\psi'(z_0)}. \tag{5.9}$$

The conformal radius is proportional to the inradius as shown by the next result.

Lemma 5.2. *For any simply connected domain $U \neq \mathbb{C}$ and any $z_0 \in U$, it holds that $\frac{1}{4}\rho(z_0, U) \leq \operatorname{dist}(z_0, \partial U) \leq \rho(z_0, U)$.*

Proof. Let $\phi : \mathbb{D} \to (\psi'(z_0)U)$ be the conformal map defined by

$$\phi(z) = \psi'(z_0)\,(\psi^{-1}(z) - z_0).$$

Then $\phi(0) = 0$ and $\phi'(0) = 1$. By Theorem 3.8, $\frac{1}{4} \leq \psi'(z_0)\operatorname{dist}(z_0, \partial U) \leq 1$. $\square$

We leave as an exercise to verify that the conformal radius of $H_t = \mathbb{H} \setminus K_t$ from z_0 is equal to

$$\rho(z_0, H_t) = \frac{2Y_t}{|g_t'(z_0)|}$$

when $t < \tau(z_0)$. The proportionality to the inradius can be written as

$$\frac{1}{2}\operatorname{dist}(z_0, \partial H_t) \leq \frac{Y_t}{|g_t'(z_0)|} \leq 2\operatorname{dist}(z_0, \partial H_t).$$

This implies that

$$\frac{1}{2}\operatorname{dist}\left(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}\right) \leq \lim_{t \nearrow \tau(z_0)} \frac{Y_t}{|g_t'(z_0)|} \leq 2\operatorname{dist}\left(z_0, \mathbb{R} \cup \overline{\gamma[0,\infty)}\right).$$

5.3.4.2 The time evolution of the conformal radius and $\arg Z_t$

For any fixed $z_0 \in \overline{\mathbb{H}}$, let $Z_t = g_t(z_0) - W_t$, $t \in [0, \tau(z_0))$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively, and as usual let $W_t = -\sqrt{\kappa}B_t$. Then from the Loewner equation it follows that

[9] Conformally covariant here means that the transformation rule under conformal maps is simple.

$$\mathrm{d}X_t = \frac{2X_t}{X_t^2+Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t \tag{5.10}$$

$$\partial_t Y_t = -\frac{2Y_t}{X_t^2+Y_t^2} \tag{5.11}$$

$$\partial_t \log|g_t'(z)| = -2\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}. \tag{5.12}$$

The last equation follows by taking derivative of the Loewner equation with respect to z.

Write using (5.11) and (5.12)

$$\partial_t \log\frac{Y_t}{|g_t'(z)|} = -4\frac{Y_t^2}{(X_t^2+Y_t^2)^2} \tag{5.13}$$

and define

$$S(t) = 4\int_0^t \frac{Y_s^2\mathrm{d}s}{(X_s^2+Y_s^2)^2} = 4\int_0^t \frac{(\sin\arg Z_t)^2}{X_s^2+Y_s^2}\mathrm{d}s. \tag{5.14}$$

Then it follow that

$$\rho(z_0,H_t) = \frac{2Y_t}{|g_t'(z_0)|} = 2y_0\exp(-S(t)) \tag{5.15}$$

where $y_0 = \mathrm{Im}\, z_0$.

Since $z \mapsto \log z$ is holomorphic, using Itô's formula for the real and imaginary parts of $\log Z_t$ gives

$$\mathrm{d}\log Z_t = (2-\kappa/2)\frac{\mathrm{d}t}{Z_t^2} + \sqrt{\kappa}\frac{\mathrm{d}B_t}{Z_t}$$

and therefore by taking real and imaginary parts we find that

$$\mathrm{d}\log|Z_t| = (2-\kappa/2)\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa}\frac{X_t}{X_t^2+Y_t^2}\mathrm{d}B_t \tag{5.16}$$

$$\mathrm{d}\arg Z_t = -(2-\kappa/2)\frac{2X_tY_t}{(X_t^2+Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa}\frac{Y_t}{X_t^2+Y_t^2}\mathrm{d}B_t. \tag{5.17}$$

Let now $\theta_t = \arg Z_t$. Then we can rewrite the previous equation as

$$\mathrm{d}\theta_t = \frac{1}{2}(\kappa-4)\sin(2\theta_t)\frac{\mathrm{d}t}{X_t^2+Y_t^2} - \sqrt{\kappa}\sin(\theta_t)\frac{\mathrm{d}B_t}{\sqrt{X_t^2+Y_t^2}}$$

5.3.4.3 A time change

Define a time change using the definitions (5.14) and $\sigma(s) = S^{-1}(s)$ in Proposition 2.6. We use the definition

$$\hat{\theta}_s = 2\theta_{\sigma(s)}$$

since it corresponds to the coordinate change from $\mathbb{H}$ to $\mathbb{D}$. After the time-changed quantities obey the time evolution

$$\mathrm{d}\hat{\theta}_s = \frac{\kappa-4}{2}\cot\left(\frac{\hat{\theta}_s}{2}\right)\mathrm{d}s + \sqrt{\kappa}\mathrm{d}\hat{B}_s \tag{5.18}$$

$$\rho(z_0, H_{\sigma(s)}) = 2y_0 e^{-s}. \tag{5.19}$$

The solution to this stochastic differential equation exists and is unique until the exist time from the interval $(0,2\pi)$

$$\hat{\tau} = \sup\{s \in \mathbb{R}_{\geq 0} : \hat{\theta}_u \in (0,2\pi) \text{ for all } u \in [0,s]\} \tag{5.20}$$

which is a stopping time. By comparing to Bessel processes, we know that if $\hat{\tau}$ is finite, then $\lim_{t\to\hat{\tau}} \hat{\theta}_t$ exists and belongs to $\{0,2\pi\}$. Since $\theta_t \in (0,\pi)$ for all $t < \tau(z_0)$, this implies that $S(t) < \hat{\tau}$ for all $t < \tau(z_0)$ and

$$S(\tau(z_0)) := \lim_{t\to\tau(z_0)} S(t) \leq \hat{\tau}. \tag{5.21}$$

We will later show that equality will indeed hold in this inequality for SLE(κ).

Furthermore, we can compare $\hat{\theta}_s$ to Bessel processes when $\hat{\theta}_s \approx 0$ or $\hat{\theta}_s \approx 2\pi$. Namely, then $\cot(\frac{\hat{\theta}_s}{2})$ is close to $\frac{2}{\hat{\theta}_s}$ or $\frac{2}{2\pi-\hat{\theta}_s}$ respectively. The corresponding Bessel process has dimension δ such that $\delta = (3\kappa-8)/\kappa$. This implies that $\hat{\tau}$ is almost surely finite for $\kappa < 8$ and almost surely infinite for $\kappa \geq 8$. The next result follows from this observation, the equation (5.15) and the equation (5.21).

Proposition 5.6. *When* $\kappa \in (0,8)$, *for any* $z \in \mathbb{H}$, *almost surely* $\mathrm{dist}(z, \overline{\gamma[0,\tau(z))}) > 0$.

5.3.5 Phase transition of $\tau(z)$ at $\kappa = 4$

We will investigate in this subsection whether $\tau(z)$ is finite or infinite, that is, whether or not z belongs to $\bigcup_{t\in\mathbb{R}_{\geq 0}} K_t$.

Proposition 5.7. *When* $\kappa > 4$, *for any* $z \in \mathbb{H}$, $\tau(z) < \infty$ *almost surely.*

Proof. Let $Z_t = g_t(z) - W_t$, $z \in \mathbb{H}$. Since $\arg Z_t \in (0,\pi)$ for all $t < \tau(z)$, Z_t can exit the set $B_R = \{z \in \mathbb{H} : |z| < R\}$ only through $\{0\} \cup \{z \in \mathbb{H} : |z| = R\}$.

Let σ_R be the exit time of $(Z_t)_{t\in\mathbb{R}_{\geq 0}}$ from B_r. Then always $\sigma_R \leq \tau(z)$. We claim that for all κ, $\sigma_R < \infty$ almost surely. To see this let $X_t = \mathrm{Re}\, Z_t$ and write

$$\mathrm{d}X_t = \sqrt{\kappa}B_t + \frac{2X_t\mathrm{d}t}{|Z_t|^2}. \tag{5.22}$$

For each $k \in \mathbb{Z}_{\geq 0}$ and $R > 0$, let $E_{k,R}$ be the event that $\min_{t\in[k,k+1]} B_t \leq B_{k+1} - R$ or $\max_{t\in[k,k+1]} B_t \geq B_{k+1} + R$. On the event $E_{k,R}$, denote by $\eta_{k,R}$ the maximal s such that $|B_s - B_{k+1}| = R$. It is fairly easy to see, for instance, by the fact that the time reversal of Brownian motion is a Brownian motion, that

$$\begin{aligned}
&\mathsf{P}\left[B_{\eta_{k,R}} - B_{k+1} = m_1 R,\ m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right] \\
=&\mathsf{P}\left[B_{\eta_{k,R}} - B_{k+1} = m_1 R \,\middle|\, E_{k,R}\right] \mathsf{P}\left[m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right] \\
=&\frac{1}{2}\,\mathsf{P}\left[m_2 X_{\eta_{k,R}} \geq 0 \,\middle|\, E_{k,R}\right]
\end{aligned} \tag{5.23}$$

for all $m_1, m_2 = \pm 1$. Therefore

$$\mathsf{P}\left[\begin{array}{c} B_{\eta_{k,R}} - B_{k+1} = -R,\ X_{\eta_{k,R}} \geq 0 \\ \text{or } B_{\eta_{k,R}} - B_{k+1} = +R,\ X_{\eta_{k,R}} \leq 0 \end{array} \middle|\, E_{k,R}\right] = \frac{1}{2}. \tag{5.24}$$

Also it is easy (by the same argument) to see that $\mathsf{P}[E_{k,R}] = \mathsf{P}[E_{1,R}] > 0$ for all k and R. Notice that the events $E_{k,R}$, $k \in \mathbb{Z}_{\geq 0}$, are independent and $\sum_{k\in\mathbb{Z}_{\geq 0}} \mathsf{P}[E_{k,R}] = \infty$, and thus by the second Borel-Cantelli lemma, see [2], $\mathsf{P}[\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{\infty} E_{k,R}] = 1$.

We claim that the conditional probability of $\sigma_R \leq k+1$ given $E_{k,R}$ is at least $1/2$. If $\tau(z) \leq k+1$, then always $\sigma_R \leq k+1$. Suppose therefore that $\tau(z) > k+1$. If either one of the events on the left of (5.24) (separated by or) occurs, then by (5.22), $|X_{k+1}| \geq |X_{k+1} - X_{\eta_{k,R}}| \geq |B_{k+1} - B_{\eta_{k,R}}| = R$ and it follows that $\sigma_R - \leq k+1$. Therefore we have shown that $\sigma < \infty$ almost surely.

Next notice that $Z_t^{1-4/\kappa}$ is a local martingale, in the sense that its real and imaginary parts are local martingales.[10] We leave as an exercise to apply Itô's formula to verify this. Let $\alpha = 1 - 4/\kappa$ and $\nu = e^{\mathrm{i}\pi(1-\alpha)/2}$. Then $h(z) = \mathrm{Im}(\nu z^\alpha)$ is a positive function on $\overline{\mathbb{H}} \setminus \{0\}$ and $h(Z_t)$ is a real-valued local martingale. It is straightforward to check that there exists a constant $c \in (0,1)$ such that $c|z|^\alpha \leq h(z) \leq |z|^\alpha$ for all $z \in \overline{\mathbb{H}}$.

If we apply the optional stopping theorem to the bounded martingale $h(Z_{t\wedge\sigma_R})$ and the stopping time σ_R, which is almost surely finite, then

$$h(z) = \mathsf{E}[h(Z_\sigma) \,|\, \sigma_R < \tau(z)]\ \mathsf{P}[\sigma_R < \tau(z)] \tag{5.25}$$

Therefore

$$c\left(\frac{|z|}{R}\right)^\alpha \leq \mathsf{P}[\sigma_R < \tau(z)] \leq c^{-1}\left(\frac{|z|}{R}\right)^\alpha \tag{5.26}$$

Thus $\mathsf{P}[\tau(z) < \infty] \geq 1 - \mathsf{P}[\sigma_R < \tau(z)] \geq 1 - c^{-1}\left(\frac{|z|}{R}\right)^\alpha$ and since $R > |z|$ is arbitrary, it follows that $\mathsf{P}[\tau(z) < \infty] = 1$. □

Proposition 5.8. *When $\kappa \in (0,4]$, for any $z \in \mathbb{H}$, $\tau(z) = \infty$ almost surely .*

Proof. The claim follows from Proposition 5.6, since $K_t = \gamma[0,t]$. □

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[10] Here and below z^α is defined as $e^{\alpha \log z}$ where the branch of log is such that $\mathrm{Im}\log z \in [0,\pi]$ for $z \in \overline{\mathbb{H}}$.

5.3.6 One-point function of SLE(κ)

5.3.6.1 The behaviour of θ_t as t tends to $\tau(z)$

Lemma 5.3 ($\lim \arg Z_t$ **for simple, transient** γ**).** *Let $z \in \mathbb{H}$ and γ be a simple curve in $\mathbb{H}$ such that $\gamma(0) \in \mathbb{H}$, $\gamma(0,\infty) \subset \mathbb{H}$, $z \notin \gamma(0,\infty)$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$. Let $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ be the Loewner chain corresponding to γ with the driving term $(W_t)_{t\in\mathbb{R}_{\geq 0}}$. Then $\lim_{t\to\infty} \arg(g_t(z) - W_t) \in \{0, 2\pi\}$.*

Proof. Let $r = |z|$ and $R > r$. By the assumptions, there exists $s \in \mathbb{R}_{\geq 0}$ such that $|\gamma(t)| \geq R$ for all $t \geq s$. By symmetry, we can suppose that z is to the right of $\gamma[0,s]$ in the sense that z can be connected by a path in $(\mathbb{H} \setminus \gamma[0,s]) \cap B(0,r)$ to the "right side" of $\gamma[0,s]$. Notice that then z is to the right of $\gamma[0,t]$ in the same sense for all $t \geq s$.

Denote by L_t the union of $\mathbb{R}_{<0}$ and the "left side" of $\gamma[0,t]$. We can write using the harmonic measure $\arg(g_t(z) - W_t) = \pi \,\mathrm{HM}(z, L_t, \mathbb{H} \setminus \gamma(0,t])$ (recall Definition 3.3 and remarks after it). By the weak Beurling estimate of the harmonic measure, $0 \leq \arg(g_t(z) - W_t) \leq C(r/R)^\alpha$ with some universal constants $C > 0$ and $\alpha > 0$. The claim follows by taking R to ∞.

The proof of the statement, that $\arg(g_t(z) - W_t)$ tends to π, when z is to the left of $\gamma[0,s]$, can be done completely symmetrically. □

Lemma 5.4 ($\lim \arg Z_t$ **when point is swallowed and not hit).** *Let $z \in \mathbb{H}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be a Loewner chain generated by a curve γ. Suppose that $\tau(z) < \infty$ and $\mathrm{dist}(z, \gamma[0,\tau(z)]) > 0$. Then $\lim_{t\to\tau(z)} \arg(g_t(z) - W_t) \in \{0, 2\pi\}$.*

Proof. The proof is very similar to the proof of Lemma 5.3. When z is to the right of $\gamma[0,\tau(z)]$, then the harmonic measure of the union of $\mathbb{R}_{<0}$ and the left-hand side of $\gamma[0,t]$ tends to zero as t tends to $\tau(z)$. Similarly when z is to the left of $\gamma[0,\tau(z)]$. □

If we combine Lemmas 5.3 and 5.4 with Propositions 5.3, 5.5, 5.6 and 5.7, we see that the inequality (5.21) is actually equality for SLE(κ). Namely, we can deduce in the following way.

- When $\kappa \in (0,4]$, γ is simple, transient and avoids the point z almost surely. Thus by Lemma 5.3, $\lim_{t\to\infty} \theta_t \in \{0,\pi\}$ and consequently $S(\infty) = \hat{\tau} < \infty$.
- When $\kappa \in (4,8)$, similarly using Lemma 5.4 it follows that $\lim_{t\to\tau(z)} \theta_t \in \{0,\pi\}$ and consequently $S(\tau(z)) = \hat{\tau} < \infty$.
- When $\kappa \in [8,\infty)$, $\tau(z) < \infty$ and $\hat{\tau} = \infty$. If it would happen that $S(\tau(z)) < \infty$, then by Lemma 5.4, $\lim_{t\to\tau(z)} \theta_t \in \{0,\pi\}$ and therefore $\hat{\tau} \leq S(\tau(z)) < \infty$, which would lead to a contradiction. Consequently, $\hat{\tau} = S(\tau(z)) = \infty$.

We have shown the next result.

Proposition 5.9. *For all κ, it holds that $\rho(z_0, (\mathbb{H} \setminus \gamma(0,\infty))_{z_0}) = 2y_0 e^{-\hat{\tau}}$ where $\hat{\tau}$ is as in* (5.20).

5.3.6.2 One-point function

Let's continue the calculation of Section 5.3.4 using Proposition 5.9. Write $z_0 = r\exp(\mathrm{i}\hat{\theta}_0/2)$, $r > 0$. Define a function, which doesn't depend on $r > 0$,

$$F(\hat{\theta}_0, u) = \mathsf{P}\left[\rho(z_0, (\mathbb{H}\setminus\gamma(0,\infty))_{z_0}) \le 2y_0 e^{-u}\right] = \mathsf{P}[\hat{\tau} \ge u].$$

Its conditional version given $\hat{\mathscr{F}}_s = \mathscr{F}_{\sigma(s)}$, can be written in the form

$$\mathsf{P}\left[\hat{\tau} \ge u \,\middle|\, \hat{\mathscr{F}}_s\right] = F(\hat{\theta}_s, u-s) \tag{5.27}$$

by the conformal Markov property of SLE(κ).

The left-hand side of (5.27) is by construction a martingale and therefore F satisfies, provided F is smooth

$$\frac{\partial F}{\partial t} = LF \tag{5.28}$$

by Itô's formula, where L is the second order differential operator

$$L = -\frac{\kappa}{2}\frac{\partial^2}{\partial x^2} - \frac{\kappa-4}{2}\cot\frac{x}{2}\frac{\partial}{\partial x}.$$

The function F satisfies the boundary conditions

$$F(x,0) = 1, \quad 0 < x < 2\pi \qquad \text{and} \qquad F(0,u) = 0 = F(2\pi,u), \quad u > 0. \tag{5.29}$$

In a suitable function space L is a self-adjoint operator. Moreover, there exists a eigenbasis $(f_k)_{k\in\mathbb{N}}$ of L such that $Lf_k = \lambda_k f_k$, $0 < \lambda_1 < \lambda_2 < \ldots$ and for each k, f_k has $k-1$ zeros on the interval $(0,2\pi)$. It is straightforward to check that

$$f(x) = \sin\left(\frac{x}{2}\right)^{\beta}$$

satisfies $Lf = \lambda f$ if and only if

$$\lambda = 1 - \frac{\kappa}{8}, \qquad \beta = \frac{8}{\kappa} - 1.$$

Since this eigenfunction is positive and thus doesn't have any zeros, it must be the eigenfunction with the smallest eigenvalue. Consequently, by an argument that we will skip, since the boundary values (5.29) are non-negative, it is possible to prove the following version of the maximum principle: there exists a constant $C > 0$ such that

$$C^{-1} f(x) e^{-\lambda u} \le F(x,u) \le C f(x) e^{-\lambda u}$$

for all $x \in [0,2\pi]$ and $u \ge 1$.

This shows that for all $z_0 \in \mathbb{H}$ and $r \in (0, \operatorname{Im} z_0)$, it holds that

$$\mathsf{P}\left[\overline{\gamma[0,\infty)} \cap \overline{B(z_0,r)} \neq \emptyset\right] \asymp \left(\frac{r}{\operatorname{Im} z_0}\right)^{\lambda} \sin\left(\arg z_0\right)^{\beta} \tag{5.30}$$

where $A \asymp B$ means that there exist positive constants c_1 and c_2 such $c_1 B \leq A \leq c_2 B$. The expression on the right-hand side of (5.30) could be called *one-point function of SLE*(κ).

5.3.7 Dimension of SLE

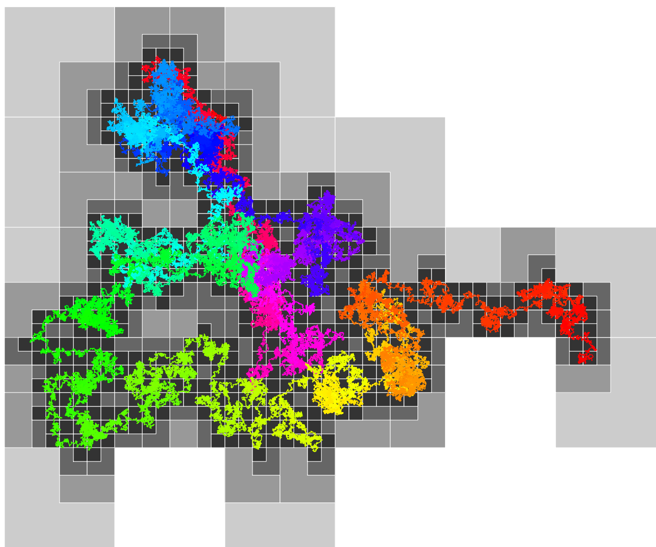

Fig. 5.4 An example of the box-counting dimension: the colorful irregular path is a sample of a planar Brownian motion and the boxes in gray a the dyadic ($\varepsilon = 2^{-n}$) boxes intersected by the path. The lightness of the gray color indicates the size of the box. The quantity $N_{2^{-n}}$ would be the number of boxes of the shade of gray corresponding to the length scale 2^{-n}.

Let us first consider a general setup where $K \subset \mathbb{C}$ is a non-empty bounded Borel set. Let N_ε be the number of sets of the form

$$[(j-1)\varepsilon, j\varepsilon] \times [(k-1)\varepsilon, k\varepsilon], \quad (j,k) \in \mathbb{Z}^2$$

intersecting with the set K. Then the *box-counting dimension* (or Minkowski dimension) of K is defined to be

$$\dim_{\mathrm{M}}(K) = \lim_{\varepsilon \searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}}$$

if the limit exists. If the limit doesn't exists we define the upper and lower box-counting dimensions as

$$\dim_{\overline{\mathrm{M}}}(K) = \limsup_{\varepsilon \searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}}, \qquad \dim_{\underline{\mathrm{M}}}(K) = \liminf_{\varepsilon \searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}},$$

respectively. It is true that the (upper and lower) box-counting dimension is not less than the Hausdorff dimension of K. Hence any upper bound for the box-counting dimension is an upper bound for the Hausdorff dimension.

In this section we will show that the following upper bound for SLE(κ), $0 < \kappa < 8$, curve γ:

$$\dim_{\overline{\mathrm{M}}}(([-1,1] \times [0,1]) \cap \gamma[0,\infty)) \leq 1 + \frac{\kappa}{8}. \tag{5.31}$$

which implies that $\dim_{\mathscr{H}}(\gamma[0,\infty)) \leq 1 + \frac{\kappa}{8}$. Remember that almost surely $\gamma[0,\infty) = \overline{\mathbb{H}}$ and thus $\dim_{\mathscr{H}}(\gamma[0,\infty)) = 2$, when $\kappa \geq 8$.[11]

By (5.30), for some $C > 0$ and $\lambda > 0$, we have a bound

$$\mathsf{P}\left[\gamma[0,\infty) \cap \overline{B(z_0,r)} \neq \emptyset\right] \leq C \left(\frac{r}{\operatorname{Im} z_0}\right)^\lambda \tag{5.32}$$

for all $z_0 \in \mathbb{H}$ and $r > 0$. If $\gamma[0,\infty)$ intersects

$$R_{j,k} = \left[(j-1)\,2^{-n}, j\,2^{-n}\right] \times \left[(k-1)\,2^{-n}, k\,2^{-n}\right],$$

then

$$\operatorname{dist}\left(\left(j - \frac{1}{2}\right) 2^{-n} + i\left(k - \frac{1}{2}\right) 2^{-n}, \gamma[0,\infty)\right) \leq 2^{-n-1/2}.$$

Hence

$$\begin{aligned} \mathsf{E} N_{2^{-n}} &= \sum_{\substack{-2^n < j \leq 2^n \\ 0 < k \leq 2^n}} \mathsf{P}\left[\gamma[0,\infty) \cap R_{j,k} \neq \emptyset\right] \leq C\,2^{-\lambda/2} \sum_{\substack{-2^n < j \leq 2^n \\ 0 < k \leq 2^n}} (k - 1/2)^{-\lambda} \\ &\leq C' 2^{(2-\lambda)n} \end{aligned}$$

where C' is a constant that depends only on C and λ.

Now by Chebyshev inequality, for each $\delta > 0$

$$\mathsf{P}\left[N_{2^{-n}} \geq 2^{(2-\lambda+\delta)n}\right] \leq C' 2^{-\delta n}.$$

Since these probabilities are summable over n, by Borel–Cantelli lemma there exist a random $n_0(\delta)$ such that

$$N_{2^{-n}} < 2^{(2-\lambda+\delta)n} \tag{5.33}$$

for $n > n_0(\delta)$. Now

[11] We didn't quite get to the statement that $\gamma[0,\infty) = \overline{\mathbb{H}}$ above. Instead, we established a weaker form $\mathsf{P}[z_0 \in \gamma[0,\infty)] = 1$ for any $z_0 \in \overline{\mathbb{H}}$.

$$\limsup_{n\to\infty} \frac{\log N_{2^{-n}}}{n\log 2} \leq 2-\lambda+\delta$$

Since $\delta > 0$ is arbitrary and $\varepsilon \mapsto N_\varepsilon$ is increasing

$$\limsup_{\varepsilon\searrow 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}} \leq 2-\lambda.$$

Hence the upper box-counting dimension is at most $2-\lambda$.

5.4 Variants of SLE

In this section, we will go through some variants of SLE. Besides the chordal SLE(κ), the most important variants of SLE are the radial SLE(κ), the dipolar SLE$[\kappa,\alpha]$ and (chordal) SLE(κ,ρ). We will also remind about the Loewner equations for the inverse maps $f_t = g_t^{-1}$ and the time-reversed maps $h_t = g_{T-t}$, because they are needed in later sections.

5.4.1 Radial SLE

5.4.1.1 Loewner equation in $\mathbb{D}$

In this subsection, we develop the Loewner theory for the unit disc $\mathbb{D}$, which is similar to the theory presented in Section 4.2 for the upper half-plane $\mathbb{H}$.

Let $K \subset \overline{\mathbb{D}}$ be a closed set such that the complement $\mathbb{D}\setminus K$ is simply connected and contains 0. We call it a *d-hull* (or $\mathbb{D}$-hull). For any d-hull K, there exists a unique conformal and onto map $g_K : \mathbb{D}\setminus K \to \mathbb{D}$ satisfying $g_K(0) = 0$ and $g_K'(0) > 0$ by the Riemann mapping theorem. The quantity $\mathrm{cap}_{\mathbb{D}}(K) = \log g_K'(0)$ is called the *d-capacity* (or $\mathbb{D}$-capacity) of K. The function g_K can be expanded around $z = 0$ as

$$g_K(z) = e^{\mathrm{cap}_{\mathbb{D}}(K)}\, z + \sum_{k=2}^{\infty} c_k z^k$$

where $c_k \in \mathbb{C}$ are some K dependent coefficients. Due to the chain rule of differentiation the d-capacity $\mathrm{cap}_{\mathbb{D}}(K)$ is additive.

Suppose, for simplicity, that the boundary of K is locally connected and thus the conformal maps we consider extend continuously to the boundary. Let $f_K = g_K^{-1}$. Define a harmonic function h on $\mathbb{D}$ by

$$h(z) = -\operatorname{Re}\log\frac{f_K(z)}{z} = -\log\left|\frac{f_K(z)}{z}\right|.$$

Notice that the holomorphic function inside log is non-zero for all $z \in \mathbb{D}$ and thus the function h is well-defined and harmonic in $\mathbb{D}$. The boundary values of h are non-negative and in fact zero outside of the set $I = g_K(K \cap \mathbb{D})$. Since h is continuous in $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$, we can use the Poisson kernel of $\mathbb{D}$ to write it as

$$h(z) = \frac{1}{2\pi} \int_I h(w) \operatorname{Re} \frac{w+z}{w-z} \, |\mathrm{d}w|. \tag{5.34}$$

The formula evaluated at $z = 0$ is the mean value property of h and it implies that

$$\operatorname{cap}_{\mathbb{D}}(K) = -\log f_K'(0) = h(0) = \frac{1}{2\pi} \int_I h(w) \, |\mathrm{d}w|.$$

In particular, the d-capacity $\operatorname{cap}_{\mathbb{D}}(K)$ is strictly positive for any non-empty K.

The identity (5.34) can be complemented with its harmonic conjugate to arrive to[12]

$$f(z) = z \exp\left(-\frac{1}{2\pi} \int_I h(w) \frac{w+z}{w-z} \, |\mathrm{d}w| \right). \tag{5.35}$$

Suppose that we have a continuously growing chain of d-hulls $(K_t)_{t \in \mathbb{R}_{\geq 0}}$. Call the parameter t time. Due to the additivity of the d-capacity it is natural to reparametrize so that $\log g_t'(0)$ is linear in time. Thus we reparametrize so that $g_t'(0) = e^t$. If the support of the increment of $(K_t)_{t \in \mathbb{R}_{\geq 0}}$ at time t is at $W_t \in \partial \mathbb{D}$, then

$$\begin{aligned} \partial_t g_t(z) &= \lim_{\delta \searrow 0} \frac{g_t(z) - g_{t-\delta}(z)}{\delta} = \lim_{\delta \searrow 0} \frac{g_t(z) - \tilde{f}_{t,\delta} \circ g_t(z)}{\delta} \\ &= g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}. \end{aligned} \tag{5.36}$$

The right sided difference quotient has the same limit, for example, by the continuity of the right-hand side of (5.36). The equation is, naturally, called *Loewner equation in* $\mathbb{D}$.

The family of hulls $(K_t)_{t \in [0,T)}$ corresponding to a solution to Loewner equation in $\mathbb{D}$ with a continuous driving term will satisfy a local growth condition analogous to the one given in Theorem 4.2. We call any of such a family a *d-Loewner chain*.

5.4.1.2 Radial SLE(κ)

Definition 5.4. *Radial SLE*(κ) *in* $(\mathbb{D}, 1, 0)$ is a stochastic d-Loewner chain $(K_t)_{t \in \mathbb{R}_{\geq 0}}$ driven by the process

$$W_t = \exp(\mathrm{i}\sqrt{\kappa} B_t)$$

where $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

---

[12] The multiplicative constant in (5.35) is fixed by evaluating both sides of (5.34) and its conjugate at 0.

Remark 5.6. As we commented in Remark 4.3, the $\mathbb{H}$-capacity parametrization is consistent with the d-capacity parametrization in the sense that choral and radial SLE(κ)'s look the same locally. We leave this argument as an exercise.

Definition 5.5. Let (U,a,w) be a triplet where U is a simply connected domain, a its boundary point and w its interior point. *Radial SLE*(κ) *in* (U,a,w) is defined as the conformal image of radial SLE in the domain $(\mathbb{D},1,0)$ under the (unique) conformal map that takes $(\mathbb{D},1,0)$ to (U,a,w).

Remark 5.7. For fixed $\kappa > 0$, the family of laws of radial SLE(κ) in (U,a,w), where (U,a,w) runs over all simply connected domains $U \neq \mathbb{C}$ as well as all $a \in \partial U$ and $w \in U$, satisfies a version of *Schramm's principle* for a domain with a marked boundary point and a marked interior point. Namely, in this version Schramm's principle, the family $(\mu^{(U,a,w)})$ of laws of random curves satisfies

- $\mu^{(U,a,w)}$ is supported on curves $\gamma(t)$ starting at a and tending to w as $t \to \infty$. The curve γ is well-described by the Loewner equation of $\mathbb{D}$.
- **Conformal invariance** (CI): $\phi_* \mu^{(U,a,w)} = \mu^{(\phi(U),\phi(a),\phi(w))}$.
- **Domain Markov property** (DMP): for any measurable set B in the space of curves, $\mu^{(U,a,w)}[\gamma|_{[t,\infty)} \in B \mid \mathscr{F}_t] = \mu^{(U\setminus\gamma([0,t]),\gamma(t),w)}[\gamma \in B]$.

Unlike in the chordal case, from the point of view of Schramm's principle, here it would be reasonable to include also a linear drift to the Brownian motion and extend the definition of radial SLE to two parameter family. It would be interpreted as chirality of the random curve as the non-zero drift would give the curve tendency to swirl around the marked interior point to the direction specified by the sign of the drift.

5.4.2 *Dipolar SLE*$[\kappa,\alpha]$

Similarly as in Section 4.2 or 5.4.1 we can develop the Loewner theory on any reference domain. We introduce it in a yet another domain, namely, in the strip $\mathbb{S}_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. The reason for this it that it leads to very interesting tools that we can utilize. We leave many details as exercises.

5.4.2.1 Loewner equation in the strip $\mathbb{S}_\pi$

A compact set $K \subset \overline{\mathbb{S}_\pi}$ whose complement in $\mathbb{S}_\pi$ is simply connected is called *s-hull*. For any s-hull K there is a unique conformal and onto map $g_K : \mathbb{S}_\pi \setminus K \to \mathbb{S}_\pi$, such that $\lim_{z\to\pm\infty}(g_K(z)-z) = \pm\mathrm{cap}_{\mathbb{S}_\pi}(K)$, where the constant $\mathrm{cap}_{\mathbb{S}_\pi}(K)$ is called *s-capacity*. Defined in this way, s-capacity is additive in composition of normalized conformal maps.

Suppose that we have a curve γ in $\mathbb{S}_\pi$ that generates a family of s-hulls $(K_t)_{t\in[0,T)}$, in an analogous sense as we have learned earlier. Since $t \mapsto \mathrm{cap}_{\mathbb{S}_\pi}(K_t)$ is non-negative and strictly increasing, it is possible to reparametrize the curve so that

$\mathrm{cap}_{\mathbb{S}_\pi}(K_t) = t$. With this *s*-capacity parametrization, the Loewner maps $g_t = g_{K_t}$ satisfy the *Loewner equation in* $\mathbb{S}_\pi$

$$\partial_t g_t(z) = \coth \frac{g_t(z) - W_t}{2}, \qquad g_0(z) = z. \tag{5.37}$$

This equation generalizes to a general family of hulls $(K_t)_{t\in[0,T)}$. Namely, being a solution to the Loewner equation is equivalent to local growth of the family of hulls. We leave as an exercise to verify that the s-capacity is positive and that the Loewner equation holds under suitable assumptions. These facts can be verified using a Poisson kernel, in the same way as we did above.

5.4.2.2 Dipolar SLE$[\kappa,\alpha]$

Definition 5.6. *Dipolar SLE*$[\kappa,\alpha]$ *in* $(\mathbb{S}_\pi, 0, +\infty, -\infty)$ is a stochastic s-Loewner chain $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ driven by the process

$$W_t = \sqrt{\kappa} B_t + \alpha t \tag{5.38}$$

where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

Definition 5.7. Let (U,a,b,c) be a quadruplet where U is a simply connected domain and a,b,c its distinct boundary points in counterclockwise order. *Dipolar SLE*$[\kappa,\alpha]$ *in* (U,a,b,c) is defined as the conformal image of dipolar SLE$[\kappa,\alpha]$ of the domain $(\mathbb{S}_\pi, 0, +\infty, -\infty)$ under the (unique) conformal map that takes $(\mathbb{S}_\pi, 0, +\infty, -\infty)$ to (U,a,b,c).

Remark 5.8. For fixed $\kappa > 0$ and $\alpha \in \mathbb{R}$, the family of laws of dipolar SLE$[\kappa,\alpha]$ in (U,a,b,c), where (U,a,b,c) runs over all simply connected domains $U \neq \mathbb{C}$ as well as all $a,b,c \in \partial U$, satisfies a version of *Schramm's principle* for a domain with distinct marked boundary points. Namely, in this version Schramm's principle, the family $(\mu^{(U,a,b,c)})$ of laws of random curves satisfies

- $\mu^{(U,a,w)}$ is supported on curves $\gamma(t)$ starting at a and tending to the boundary arc between b and c as $t \to \infty$. The curve γ is well-described by the Loewner equation of $\mathbb{S}_\pi$.
- **Conformal invariance** (CI): $\phi_* \mu^{(U,a,b,c)} = \mu^{(\phi(U),\phi(a),\phi(b),\phi(c))}$.
- **Domain Markov property** (DMP): for any measurable set B in the space of curves, $\mu^{(U,a,b,c)}[\,\gamma|_{[t,\infty)} \in B \mid \mathscr{F}_t\,] = \mu^{(U\setminus\gamma([0,t]),\gamma(t),b,c)}[\gamma \in B]$.

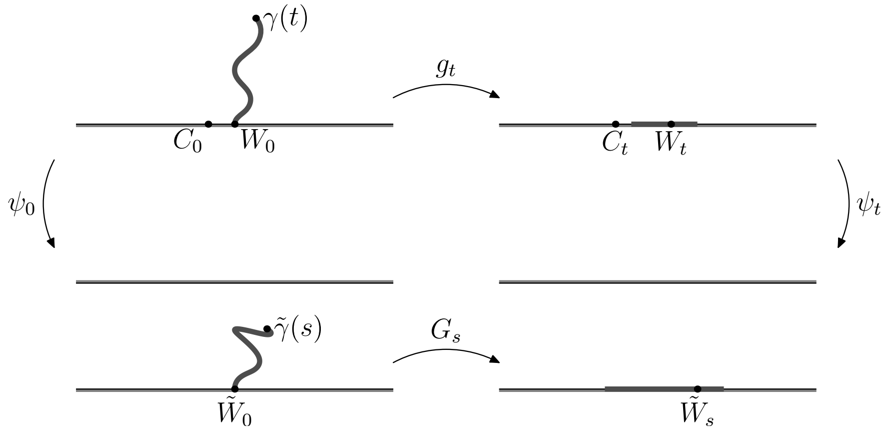

Fig. 5.5 Coordinate transform of a SLE-type process from $\mathbb{H}$ to $\mathbb{S}_\pi$

5.4.3 Coordinate changes and SLE(κ,ρ)

5.4.3.1 Coordinate transform of SLE(κ) from $\mathbb{H}$ to $\mathbb{S}_\pi$

Let's consider SLE(κ) on $\mathbb{H}$ and a conformal transformation from $\mathbb{H}$ onto the strip $\mathbb{S}_\pi = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$ Let $c < 0$. The unique conformal map ψ_0 from with $\psi_0(0) = 0$, $\psi_0(c) = -\infty$ and $\psi_0(\infty) = +\infty$ is given by

$$\psi_0(z) = \log(z-c) - \log|c|.$$

The point c evolves under the SLE(κ) flow as $C_t = g_t(c)$. Let $\psi_t(z) = \log(z - C_t) + \delta(t)$ where $\delta(t)$ is a constant such that the map

$$\hat g_t = \psi_t \circ g_t \circ \psi_0^{-1} \tag{5.39}$$

satisfies the normalization introduced in Section 5.4.2.1. After a calculation we find that $-\frac{1}{2}\log g_t'(c) - \log|c|$.

Consequently, the s-capacity $S(t)$ of the s-hull $\psi_0(K_t)$ can be written as

$$S(t) = -\frac{1}{2}\log g_t'(c) = \int_0^t \frac{\mathrm{d}u}{(W_t - C_t)^2},$$

and the driving term transforms to

$$\hat W_t = \log(W_t - C_t) + S(t) - \log|c| \ \in \mathbb{R} \subset \partial\mathbb{S}_\pi.$$

Define a time-change $\sigma = S^{-1}$ and set $\tilde g_s = \hat g_{\sigma(s)}$ and $\tilde W_s = \hat W_{\sigma(s)}$. A straightforward calculation shows that $\tilde g_s$ satisfies the Loewner equation of the strip $\mathbb{S}_\pi$

$$\partial_s \tilde{g}_s(z) = \coth \frac{\tilde{g}_s(z) - \tilde{W}_s}{2}, \qquad \tilde{g}_0(z) = z$$

And if the driving term in the upper half-plane is a Brownian motion then the driving term of the strip is a Brownian motion with a drift. See Table 5.1 for more details.

	$\mathbb{H}$	$\mathbb{S}_\pi$
Normalization	$g_t(z) = z + \frac{2t}{z} + \ldots, z \to \infty$	$\tilde{g}_s(z) = \begin{cases} z - s + o(1), & z \to -\infty \\ z + s + o(1), & z \to +\infty \end{cases}$
Loewner equation	$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$	$\partial_s \tilde{g}_s(z) = \coth \frac{\tilde{g}_s(z) - \tilde{W}_s}{2}$
Driving term	$W_t \in \mathbb{R}$	$\tilde{W}_s \in \mathbb{R}$
Chordal SLE(κ)	$W_t = \sqrt{\kappa} B_t$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha_0(\kappa)s$
SLE(κ, ρ) and Dipolar SLE[κ, α]	$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}$	$\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s + \alpha s$
Relations between parameters	$\alpha = \rho + 3 - \frac{\kappa}{2}, \qquad \alpha_0(\kappa) = 3 - \frac{\kappa}{2}$	

Table 5.1 A comparison between SLE in $\mathbb{H}$ and in $\mathbb{S}_\pi$. For clarity we use separate notations for quantities in $\mathbb{H}$ and in $\mathbb{S}_\pi$.

5.4.3.2 Coordinate transform of dipolar SLE$[\kappa, \alpha]$ and the definition of SLE(κ, ρ)

Definition 5.8. Let $\kappa \geq 0$ and $\rho \in \mathbb{R}$. Let $w_0, c_0 \in \mathbb{R}$ with $w_0 \neq c_0$ and let $(W_t, C_t)_{t \in [0, \tau(c_0))}$ be the solution to the system of stochastic differential equations

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \dfrac{\rho}{W_t - C_t}\mathrm{d}t \\ \mathrm{d}C_t = \dfrac{2}{C_t - W_t}\mathrm{d}t \end{cases}, \qquad \begin{cases} W_0 = w_0 \\ C_0 = c_0 \end{cases} \tag{5.40}$$

which exists for $t \in [0, \tau(c_0))$ where $\tau(c_0) = \sup\{t \in \mathbb{R}_{\geq 0} : \inf_{s\in[0,t]} |W_t - C_t| > 0\}$. Then the Loewner chain $(g_t, K_t)_{t\in[0,\tau(c_0))}$ with the driving process $(W_t)_{t\in[0,\tau(c_0))}$ is called *SLE*(κ, ρ).

Remark 5.9. The chordal SLE(κ) is a special case SLE$(\kappa, 0)$ of this definition.

Remark 5.10. It is possible to construct SLE(κ, ρ) using a Bessel process. This construction is especially useful when we want to consider the process beyond $\tau(c_0)$, which we don't do in this text, but we'll give this construction here. Let $w_0, c_0 \in \mathbb{R}$ with $w_0 \neq c_0$ and let $\eta = \operatorname{sgn}(w_0 - c_0)$. Let D_t be the Bessel process (with an unusual time-parametrization)

$$\mathrm{d}D_t = \frac{\rho+2}{D_t}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}\tilde{B}_t, \qquad D_0 = |w_0 - c_0|.$$

Define

$$C_t = c_0 - 2\eta \int_0^t \frac{\mathrm{d}u}{D_u}, \qquad W_t = C_t + \eta D_t.$$

Then they satisfy (5.40) with $B_t = \eta \tilde{B}_t$.

Definition 5.9. Let (U, a, b, c) be a quadruplet where U is a simply connected domain and a, b, c its distinct boundary points in counterclockwise order. *SLE*(κ, ρ) *in* (U, a, b, c) is defined as the conformal image of SLE(κ, ρ) of the domain $(\mathbb{H}, 0, \infty, -1)$ under the (unique) conformal map that takes $(\mathbb{H}, 0, \infty, -1)$ to (U, a, b, c).

The above coordinate change calculation shows the following result, which is also in the last row of Table 5.1.

Lemma 5.5. *When $\alpha = \rho + 3 - \frac{\kappa}{2}$, the dipolar SLE$[\kappa, \alpha]$ and SLE(κ, ρ) in the domain (U, a, b, c) considered up to time of disconnection are equal (up to a time change) in distribution.*

5.4.4 Special parameters values of SLE(κ, ρ)

5.4.4.1 SLE$(\kappa, (\kappa - 6)/2)$ is symmetric

Denote the reflection with respect to the y-axis by

$$m(z) = -\bar{z}. \tag{5.41}$$

Then m is an antiholomorphic map from $\mathbb{C}$ onto itself. Since the process $\tilde{W}_s = \sqrt{\kappa}\tilde{B}_s$ is invariant under $\tilde{W}_s \mapsto -\tilde{W}_s$, SLE$(\kappa, (\kappa - 6)/2)$ on $\mathbb{S}_\pi$ is invariant under m and for fixed $\kappa > 0$, it is the unique SLE(κ, ρ) process with this property. We say that SLE$(\kappa, (\kappa - 6)/2)$ on $\mathbb{S}_\pi$ is *symmetric*.

Suppose that we know that some discrete random curve arising from statistical physics converges to SLE(κ) as the mesh goes to zero. For example, suppose we know that the interface of Ising model with boundary conditions changing at two marked points (boundary conditions are $+$ spins on one arc and $-$ spins on the other arc) converges to SLE(3). Can we conclude something about the scaling limit for other boundary conditions? If we consider the Ising model with three marked points $a,b,c \in \partial U$ (in counterclockwise order), instead, and boundary conditions are set to be $-$ on the arc ab, $+$ on the arc ca and free on the arc bc, then by Schramm's principle we expect that the scaling limit of the interface starting from the point a should be SLE$(3,\rho)$ process. And since the law of that interface is invariant under flipping all the spins $\sigma \to -\sigma$, the scaling limit should be symmetric on $\mathbb{S}_\pi$ and hence it should be SLE$(3,-3/2)$.

5.4.4.2 SLE(6) satisfies locality

Consider following map

$$\psi = m \circ \phi^{-1} \circ m \circ \phi \tag{5.42}$$

where m is as in (5.41). Under those maps SLE(κ) is transformed as

$$\begin{aligned}(\mathbb{H}, \kappa = 6, \rho = 0) &\xrightarrow{\phi} (\mathbb{S}_\pi, \kappa = 6, \alpha = 0) \xrightarrow{m} (\mathbb{S}_\pi, \kappa = 6, \alpha = 0) \\ &\xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa = 6, \rho = 0) \xrightarrow{m} (\mathbb{H}, \kappa = 6, \rho = 0)\,.\end{aligned}$$

On the other hand ψ is a holomorphic and bijective self map of $\mathbb{H}$ with $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\psi(c) = |c|$. Hence $\psi(z) = \frac{|c|z}{z-c}$. Therefore SLE$(6)$ has the following *locality* property: the image of SLE(6) under any conformal self-map of $\mathbb{H}$ is again (a time-change of) SLE(6). If $\psi : \mathbb{H} \to \mathbb{H}$ is this Möbius map, then we consider the first process until it disconnects $\psi^{-1}(\infty)$ from ∞ and the second one until it disconnects $\psi(\infty)$ from ∞. Actually SLE(6) has even stronger locality property because SLE(6) sent from 0 is invariant up to a time-change under any conformal transformation defined in a neighborhood of 0 such that it maps a neighborhood of 0 in $\mathbb{R}$ into $\mathbb{R}$.

5.4.4.3 SLE$(\kappa,\kappa-6)$ is target independent

For other values of κ, the argument of the section 5.4.4.2 gives that if $(K_t)_{t\in[0,\tau(c))}$ is a chordal SLE(κ) stopped at the time $\tau(c)$ then $(\psi(K_t))_{t\in[0,\tau(c))}$ is a time-change of the SLE$(\kappa,\kappa-6)$ process stopped at the time when the process disconnects $|c|$ from ∞. Namely, under the map ψ of the form (5.42) the processes are transformed in the following way:

$$\begin{aligned}(\mathbb{H}, \kappa, \rho = 0) &\xrightarrow{\phi} (\mathbb{S}_\pi, \kappa, \alpha = 3 - \kappa/2) \xrightarrow{m} (\mathbb{S}_\pi, \kappa, \alpha = \kappa/2 - 3) \\ &\xrightarrow{\phi^{-1}} (\mathbb{H}, \kappa, \rho = \kappa - 6) \xrightarrow{m} (\mathbb{H}, \kappa, \rho = \kappa - 6)\,.\end{aligned}$$

5.4.5 *The reverse flow of the chordal SLE*

5.4.5.1 The Loewner equation of inverse $f_t = g_t^{-1}$

Let us go back to the Loewner equation in $\mathbb{H}$. It is customary to denote the inverse of g_t by $f_t = g_t^{-1}$. By simply differentiating the expression $f_t(g_t(w)) = w$ on both sides, we arrive to the identity

$$(\partial_t f_t)(g_t(w)) + f_t'(g_t(w))\,\partial_t g_t(w) = 0.$$

Replacing $g_t(w)$ by z this gives *Loewner equation of f_t*

$$\partial_t f_t(z) = -\frac{2 f_t'(z)}{z - W_t}. \tag{5.43}$$

5.4.5.2 The time-reversed Loewner equation

Let us introduce the *reverse Loewner equation*

$$\partial_t h_t(z) = -\frac{2}{h_t(z) - V_t}, \qquad h_0(z) = z. \tag{5.44}$$

The next result was shown in the proof of Proposition 4.1.

Lemma 5.6. *Let $h_t(z)$ be the solution of* (5.44) *where $(V_t)_{t\in\mathbb{R}_{\geq 0}}$ is continuous. Then the solution is well-defined for all $t \in \mathbb{R}_{\geq 0}$. More over $t \mapsto \operatorname{Im} h_t(z)$ is strictly increasing.*

The following lemma gives the significance of the reverse flow.

Lemma 5.7. *Let $(W_t)_{t\in[0,T]}$ be continuous and $\tilde{V}_t = W_{T-t}$, $t \in [0,T]$. Let $\tilde{h}_t(z)$ be the reverse Loewner flow with the driving term $(\tilde{V}_t)_{t\in[0,T]}$ and the $f_t(z)$ be the inverse Loewner flow with the driving term $(W_t)_{t\in[0,T]}$. Then the functions $z \mapsto f_T(z)$ are $z \mapsto \tilde{h}_T(z)$ equal.*

Proof. We will show that $g_T \circ \tilde{h}_T(z) = z$ for all $z \in \mathbb{H}$.

Fix $z \in \mathbb{H}$ and let $\zeta_t = \tilde{h}_{T-t}(z)$ for all $t \in [0,T]$. Then $\zeta_0 = \tilde{h}_T(z)$ and

$$\dot{\zeta}_t = \frac{2}{\tilde{h}_{T-t}(z) - \tilde{V}_{T-t}} = \frac{2}{\zeta_t - W_t}.$$

Hence $\zeta_t = g_t(\zeta_0)$ for all $t \in [0,T]$. In particular, $z = \zeta_T = g_T(\tilde{h}_T(z))$. □

Continue the setup of the previous lemma and set $h_t(z) = \tilde{h}_t(z + W_T) - W_T$. Then by a straightforward calculation, h_t satisfies the reverse Loewner equation with a driving term $V_t = W_{T-t} - W_t$. This observation leads to the following "symmetry" of the chordal SLE(κ).

Lemma 5.8. *Let $h_t(z)$ be the solution of* (5.44) *for $V_t = \sqrt{\kappa}B_t$ and let $f_t(z)$ be the solution of* (5.43) *for $W_t = \sqrt{\kappa}B_t$. Then for any $t \in \mathbb{R}_{\geq 0}$, the functions $z \mapsto f_t(z+W_t) - W_t$ and $z \mapsto h_t(z)$ have the same distribution. In particular, $f_t'(z+W_t)$ has the same distribution as $h_t'(z)$.*

Remark 5.11. This result holds only for a single time instant. It is not true that the joint law of $(f_t(z+W_t)-W_t)_{t\in\mathbb{R}_{\geq 0}}$, is the same as the joint law of $(h_t(z))_{t\in\mathbb{R}_{\geq 0}}$.

5.4.5.3 A lemma on the continuity of the solution of Loewner equation

Lemma 5.9. *For each $\delta > 0$ and $T > 0$ there exists a constant $C(T,\delta)$ such that the following holds. Let $h_k(t,z)$, $k = 1,2$ be the solutions of* (5.44) *with the continuous driving terms $(W_k(t))_{t\in[0,T]}$, $k = 1,2$, respectively. Then they satisfy*

$$|h_1(T,z_1) - h_2(T,z_2)| \leq C(T,\delta)(\|W_1 - W_2\|_{\infty,[0,T]} + |z_1 - z_2|)$$

for any z_1, z_2 such that $\operatorname{Im} z_k > \delta > 0$.

Proof. Fix $\delta > 0$, $T > 0$ and $z_k \in \mathbb{H}$, $k = 1,2$, such that $\operatorname{Im} z_k > \delta$, $k = 1,2$. Let $h_k(t,z)$, $k = 1,2$ be the solutions of (5.44) with the continuous driving terms $(W_k(t))_{t\in[0,T]}$, which we also consider to be fixed. Write

$$\psi(t) = h_1(T,z_1) - h_2(T,z_2).$$

Then

$$\partial_t \psi(t) = \zeta(t)\,(\psi(t) - D(t))$$

where $\zeta(t) = 2/((h_1(t,z_1) - W_1(t))(h_2(t,z_2) - W_2(t)))$ and $D(t) = W_1(t) - W_2(t)$.

Since $\operatorname{Im} z_k > \delta$, $k = 1,2$, it follows that $|\zeta(t)| \leq 2\delta^{-2}$. Using an integrating factor we can write

$$\partial_t \left(e^{-\int_0^t \zeta(s)\,\mathrm{d}s}\, \psi(t) \right) = -\zeta(t) e^{-\int_0^t \zeta(s)\,\mathrm{d}s} D(t)$$

And hence

$$e^{-\int_0^t \zeta(s)\,\mathrm{d}s}\, \psi(t) - \psi(0) = -\int_0^t \zeta(u) e^{-\int_0^u \zeta(s)\,\mathrm{d}s} D(u)\mathrm{d}u$$

which we can write as

$$\psi(t) = e^{\int_0^t \zeta(s)\,\mathrm{d}s}\, \psi(0) - \int_0^t \zeta(u) e^{\int_u^t \zeta(s)\,\mathrm{d}s} D(u)\mathrm{d}u.$$

Thus using the triangle inequality and the upper bound for $|\zeta(t)|$

$$|\psi(t)| \leq e^{\frac{2t}{\delta^2}}\, |\psi(0)| + \frac{2t}{\delta^2} e^{\frac{2t}{\delta^2}} \|D\|_{\infty,[0,T]}.$$

This gives the claim. ◻

5.5 Moments of the derivative of the Loewner map of SLE(κ)

We will present in this section the auxiliary results needed for the proof of Theorem 5.2. The function $\gamma(t) = \lim_{\varepsilon \searrow 0} f_t(W(t) + i\varepsilon)$ is called the *trace* of the Loewner chain. It is useful to define

$$\tilde{f}_t(z) = f_t(z + W_t). \tag{5.45}$$

The goal of this section is to have good bounds for $|\tilde{f}_t'(iy)|$, $t \in [0,1], y \in (0,1]$. The proof of the theorem will be given in Section 6.2 below.

5.5.1 *The setup and useful formulas*

Let's deal with both the forward and reverse Schramm–Loewner evolution by fixing $\nu = \pm 1$ and letting $h_t(z)$ be the solution of the following equation

$$\partial_t h_t(z) = \nu \frac{2}{h_t(z) - W_t}, \qquad h_0(z) = z$$

where $W_t = -\sqrt{\kappa} B_t$. For fixed $z_0 = x_0 + iy_0 \in \mathbb{H}$, let $Z_t = h_t(z_0) - W_t$ and let X_t and Y_t be the real and imaginary parts of Z_t, respectively.

Let's list some useful formulas which are mostly familiar from Section 5.3.

$$\mathrm{d}X_t = 2\nu \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}t + \sqrt{\kappa}\mathrm{d}B_t, \qquad \partial_t Y_t = -2\nu \frac{Y_t}{X_t^2 + Y_t^2},$$
$$\partial_t |h_t'(z_0)| = -2\nu |h_t'(z_0)| \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}, \qquad \partial_t \frac{|h_t'(z_0)|}{Y_t} = 4\nu \frac{|h_t'(z_0)|}{Y_t} \frac{Y_t^2}{(X_t^2 + Y_t^2)^2},$$
$$\mathrm{d}\arg Z_t = (\kappa - 4\nu) \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{Y_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,$$
$$\mathrm{d}\log|Z_t| = -\frac{1}{2}(\kappa - 4\nu) \frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t + \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t,$$
$$\mathrm{d}\sin\arg Z_t = (\sin\arg Z_t) \left[\frac{(\kappa - 4\nu) X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2 + Y_t^2)^2}\mathrm{d}t - \sqrt{\kappa} \frac{X_t}{X_t^2 + Y_t^2}\mathrm{d}B_t \right].$$

We leave as an exercise to verify these formulas. Now we fix $\nu = -1$. Then all the processes above are well-defined for all $t \in \mathbb{R}_{\geq 0}$.

5.5.2 *The martingale argument for the expectation of the moments*

Let $p, q, r \in \mathbb{R}$ and define

$$M_t = |h_t'(z_0)|^p \, Y_t^q \, (\sin \arg Z_t)^{-2r}.$$

By Itô's formula,

$$\begin{aligned} \mathrm{d}M_t = M_t \Bigg(& 2p \frac{X_t^2 - Y_t^2}{(X_t^2+Y_t^2)^2} + 2q \frac{Y_t}{X_t^2+Y_t^2} - 2r \frac{(\kappa+4)X_t^2 - \frac{\kappa}{2} Y_t^2}{(X_t^2+Y_t^2)^2} \\ & + r(2r+1) \frac{\kappa X_t^2}{(X_t^2+Y_t^2)^2} \Bigg) \mathrm{d}t - 2r\sqrt{\kappa} \frac{X_t}{X_t^2+Y_t^2} M_t \mathrm{d}B_t \end{aligned} \tag{5.46}$$

Therefore M_t is a local martingale if and only if

$$q = p - \frac{\kappa}{2} r, \qquad r^2 - \left(1 + \frac{4}{\kappa}\right) r + \frac{2}{\kappa} p = 0$$

and in that case (5.46) simplifies to

$$\mathrm{d}M_t = -2r\sqrt{\kappa} \frac{X_t}{X_t^2+Y_t^2} M_t \, \mathrm{d}B_t.$$

Next we define a time change that simplifies the above formula. Let

$$S(t) = \int_0^t \frac{\mathrm{d}u}{X_u^2+Y_u^2}, \qquad \sigma(s) = S^{-1}(s) \tag{5.47}$$

and $\hat{\mathscr{F}}_s = \mathscr{F}_{\sigma(s)}$ and use Proposition 2.6. Then

$$\hat{B}_s = \int_0^{\sigma(s)} \frac{\mathrm{d}B_u}{\sqrt{X_u^2+Y_u^2}}$$

is a standard one-dimensional Brownian motion with respect to the filtration $(\hat{\mathscr{F}}_s)_{s \in \mathbb{R}_{\geq 0}}$. Denote the time-changed processes by

$$\hat{Z}_s = Z_{\sigma(s)}, \qquad \hat{X}_s = X_{\sigma(s)}, \qquad \hat{Y}_s = Y_{\sigma(s)}, \qquad \hat{h}_s(z_0) = h_{\sigma(s)}(z_0).$$

Notice that the equations

$$\partial_s \hat{Y}_s = 2\hat{Y}_s, \qquad \partial_s \frac{|\hat{h}_s'(z_0)|}{\hat{Y}_s} = -4 \frac{|\hat{h}_s'(z_0)|}{\hat{Y}_s} (\sin \arg \hat{Z}_s)^2$$

hold and therefore

$$\hat{Y}_s = y_0 e^{2s} \tag{5.48}$$

$$|\hat{h}_s'(z_0)| = \exp\left(2s - 4\int_0^s (\sin \arg \hat{Z}_u)^2 \mathrm{d}u\right). \tag{5.49}$$

Hence (5.48) shows that the time change (5.47) is such that $\hat{Y}_s$ is deterministically exponentially increasing. The equation (5.49) implies that

$$e^{-2s'} \leq \frac{|\hat{h}'_{s+s'}(z_0)|}{|\hat{h}'_s(z_0)|} \leq e^{2s'}. \tag{5.50}$$

Observe also that

$$Y_t \leq \sqrt{y_0^2 + 4t} \tag{5.51}$$

This shows that $y_0 e^{2s} \leq \sqrt{y_0^2 + 4\sigma(s)}$ and hence $\sigma(s) \to \infty$ as $s \to \infty$.

Under this time-change, the local martingale $\hat{M}_s = M_{\sigma(s)}$ satisfies

$$\mathrm{d}\hat{M}_s = -2r\sqrt{\kappa}\,(\cos\arg\hat{Z}_s)\,\hat{M}_s\,\mathrm{d}B_s.$$

It is not hard to show that $(\hat{M}_s)_{s\in\mathbb{R}_{\geq 0}}$ is a martingale.

Lemma 5.10. *Let N_0 be a constant and let $(N_t)_{t\in\mathbb{R}_{\geq 0}}$ be a local martingale with*

$$N_t = N_0 + \int_0^t A_s N_s \mathrm{d}B_s.$$

If for every $t > 0$ there is a constant $c(t)$ such that $|A_s| \leq c(t)$ for all $s \in [0,t]$, then N_t is a martingale.

Proof. Let $M_t = N_t - N_0$. Then

$$M_t = \int_0^t (A_s M_s + A_s N_0)\mathrm{d}B_s.$$

Let $n \in \mathbb{N}$ and define $T = \inf\{t \in \mathbb{R}_{\geq 0} : \langle M\rangle_t = n\}$. Then $M_{t\wedge T}$ is an Itô integral with a $\mathscr{L}^2$ integrand. Define $f(t) = \mathsf{E}[M^2_{t\wedge T}]$. By the Itô isometry

$$f(t) = \mathsf{E}\left[\int_0^t (A_s M_s + A_s N_0)^2 \mathbb{1}_{s\leq T}\mathrm{d}s\right]$$

Therefore for any $t' \in [0,t]$

$$f(t') \leq 2c(t)^2 N_0^2 t' + 2c(t)^2 \int_0^{t'} f(s)\,\mathrm{d}s \leq \tilde{c}(t)\,t' + \tilde{c}(t)\int_0^{t'} f(s)\,\mathrm{d}s \tag{5.52}$$

where $\tilde{c}(t) = 2c(t)^2 \max\{1, N_0^2\}$. This implies that $f(t') < \exp(2\tilde{c}(t)t')$ because no $t' \in [0,t]$ can be the smallest s such that $f(s) \geq \exp(2\tilde{c}(t)s)$ by (5.52). Therefore

$$\mathsf{E}[\langle N\rangle_{t\wedge T}] = \mathsf{E}[\langle M\rangle_{t\wedge T}] = f(t) < \exp(2\tilde{c}(t)t)$$

Taking $n \to \infty$ we get by the monotone convergence theorem that $\mathsf{E}[\langle N\rangle_t] \leq \exp(2\tilde{c}(t)t)$. This shows that the integrand $A_t N_t$ is in $\mathscr{L}^2$ and hence by the construction of the Itô integral, N_t is a martingale. □

The next theorem is the main result of this section.

Theorem 5.5. *Let $(p,r) \in \mathbb{R}^2$ be a solution of the equation*

$$r^2 - \left(1 + \frac{4}{\kappa}\right) r + \frac{2}{\kappa} p = 0$$

Then

$$\hat{M}_s = |\hat{h}_s'(z_0)|^p \, \hat{Y}_s^{p - \frac{\kappa}{2} r} \, (\sin \arg \hat{Z}_s)^{-2r}$$

is a martingale and

$$\mathsf{E}\left[|\hat{h}_s'(z_0)|^p \, (\sin \arg \hat{Z}_s)^{-2r}\right] = e^{-2s\left(p - \frac{\kappa}{2} r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Furthermore, if $r \geq 0$ and $p \geq 0$, then

$$\mathsf{P}\left[|\hat{h}_s'(z_0)| \geq \lambda\right] \leq \lambda^{-p} e^{-2s\left(p - \frac{\kappa}{2} r\right)} \left(\frac{y_0}{|z_0|}\right)^{-2r}.$$

Proof. We have already shown the first claim. For the second one notice that

$$\hat{M}_s = y_0^{p - \frac{\kappa}{2} r} \, e^{2s\left(p - \frac{\kappa}{2} r\right)} \, |\hat{h}_s'(z_0)|^p \, (\sin \arg \hat{Z}_s)^{-2r}.$$

If $r \geq 0$, then $(\sin \arg \hat{Z}_s)^{-2r} \geq 1$ and the last claim follows from the Chebyshev inequality. □

Corollary 5.1. *Let $\tilde{f}_t$ be defined as in (5.45) for SLE(κ). For every $0 \leq r \leq 1 + 4/\kappa$, there is a constant $c = c(\kappa, r) < \infty$ such that for all $0 \leq t \leq 1$, $0 < y_0 \leq 1$, $e^6 \leq \lambda \leq y_0^{-1}$,*

$$\mathsf{P}\left[|\tilde{f}_t'(z_0)| \geq \lambda\right] \leq c \lambda^{-p} \left(\frac{y_0}{|z_0|}\right)^{-2r} \delta(y_0, \lambda). \tag{5.53}$$

Here $p = \frac{\kappa}{2}\left(\left(1 + \frac{4}{\kappa}\right) r - r^2\right) \geq 0$ and

$$\delta(y_0, \lambda) = \begin{cases} \lambda^{-p + \frac{\kappa}{2} r} & \text{when } p - \frac{\kappa}{2} r > 0 \\ 1 + \log \frac{1}{\lambda y_0} & \text{when } p - \frac{\kappa}{2} r = 0 \\ y_0^{p - \frac{\kappa}{2} r} & \text{when } p - \frac{\kappa}{2} r < 0 \end{cases}.$$

Proof. Since $\tilde{f}_t'$ and h_t' have the same distribution, it is enough to show (5.53) when $\tilde{f}_t'$ is replaced by h_t'. Notice first that $Y_t \leq \sqrt{y_0^2 + 4t} \leq \sqrt{5}$. Therefore

$$\mathsf{P}\left[|h_t'(z_0)| \geq \lambda\right] \leq \mathsf{P}\left[\sup_{0 \leq s \leq T} |\hat{h}_s'(z_0)| \geq \lambda\right]$$

where $T = (\log(\sqrt{5}/y_0))/2$. Next notice that by (5.50), $|\hat{h}_{s+s'}'(z_0)| \leq e^{2s'} |\hat{h}_s'(z_0)|$ and therefore

$$\mathsf{P}\left[\sup_{0\le s\le T}|\hat{h}_s'(z_0)|\ge\lambda\right]\le\sum_{j=0}^{\lfloor T\rfloor}\mathsf{P}\left[|\hat{h}_j'(z_0)|\ge e^{-2}\lambda\right]$$

Also by (5.50), $|\hat{h}_s'(z_0)|\le e^{2s}$ and therefore

$$\begin{aligned}\mathsf{P}\left[\sup_{0\le s\le T}|\hat{h}_s'(z_0)|\ge\lambda\right]&\le\sum_{j=\lceil\log(\lambda)/2-1\rceil}^{\lfloor T\rfloor}\mathsf{P}\left[|\hat{h}_j'(z_0)|\ge e^{-2}\lambda\right]\\&\le e^{2p}\lambda^{-p}\left(\frac{y_0}{|z_0|}\right)^{-2r}\sum_{j=\lceil\log(\lambda)/2-1\rceil}^{\lfloor T\rfloor}e^{-2j\left(p-\frac{\kappa}{2}r\right)}\\&\le c\,\lambda^{-p}\left(\frac{y_0}{|z_0|}\right)^{-2r}\delta(y_0,\lambda).\end{aligned}$$

Here we use that $\sum_{k=n}^{m}\beta^k\le\beta^n/(1-\beta)$ when $0<\beta<1$ and similar bounds for $\beta=1$ and $\beta>1$. □

Next we apply the previous result and optimize over the parameters for fixed κ. Let's parametrize p in terms of r as

$$p(r)=\frac{\kappa}{2}\left(\left(1+\frac{4}{\kappa}\right)r-r^2\right)$$

and study the quantity

$$\alpha(r)=2p(r)-\frac{\kappa}{2}r=\kappa r\left(\left(\frac{1}{2}+\frac{4}{\kappa}\right)-r\right).$$

Notice that $\alpha(r)$ is maximized by $r_0=1/4+2/\kappa$ and

$$\alpha(r_0)=\kappa\left(\frac{1}{4}+\frac{2}{\kappa}\right)^2=\frac{\kappa}{16}+1+\frac{4}{\kappa}\ge 2$$

and $\alpha(r_0)=2$ only if $\kappa=8$.

Let $\kappa\ne 8$ and set $p_0=p(r_0)$. Then $p_0>\kappa r_0/2$ if $\kappa<8$ and $p_0<\kappa r_0/2$ if $\kappa>8$. Let $\theta\in(0,1-\frac{2}{2p_0-\kappa r_0/2})$. Let $t\in[0,1]$ and $n\in\mathbb{N}$. By the estimate (5.53) for $r=r_0$ and $p=p_0$, we have that for large enough n

$$\begin{aligned}\mathsf{P}\left[\left|\tilde{f}_t'(i2^{-n})\right|\ge 2^{n(1-\theta)}\right]&\le c\,2^{-p_0(1-\theta)n}\,\delta\left(2^{-n},2^{n(1-\theta)}\right)\\&=c\,2^{-p_0(1-\theta)n}\times\begin{cases}2^{-(1-\theta)(p_0-\frac{\kappa}{2}r_0)n} & \text{when } \kappa<8\\ 2^{-(p_0-\frac{\kappa}{2}r_0)n} & \text{when } \kappa>8\end{cases}\\&\le c\,2^{-(1-\theta)(2p_0-\frac{\kappa}{2}r_0)n}=c2^{-(2+\varepsilon)n}\end{aligned}$$

for some $\varepsilon>0$. Let

$$\mathscr{D}_{2n} = \{k2^{-2n} : k = 0, 1, 2, \ldots, 2^{2n}\}$$

which is the dyadic partitioning of $[0,1]$ into intervals of length 2^{-2n}. Then

$$\sum_{n\in\mathbb{N}} \sum_{t\in\mathscr{D}_{2n}} \mathsf{P}\left[\left|\tilde{f}_t'(i2^{-n})\right| \geq 2^{n(1-\theta)}\right] < \infty$$

and hence the Borel–Cantelli lemma implies the following result.

Proposition 5.10. *Let $\tilde{f}_t$ be defined as in* (5.45) *for SLE(κ). For each $\kappa \neq 8$ there exists $\theta_0(\kappa) > 0$ such that the following holds: For any $\theta \in (0, \theta_0(\kappa))$, there exists a random variable C such that $C < \infty$ almost surely and*

$$\left|\tilde{f}_t'(i2^{-n})\right| \leq C\,2^{n(1-\theta)} \tag{5.54}$$

for any $t \in \mathscr{D}_{2n}$ and for any $n \in \mathbb{N}$.

Remark 5.12. By the above, we can choose

$$\theta_0(\kappa) = \frac{\frac{\kappa}{16} + \frac{4}{\kappa} - 1}{\frac{\kappa}{16} + \frac{4}{\kappa} + 1}.$$

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