Chapter 4
Loewner equation

4.1 Conformal maps of the upper half-plane

4.1.1 Half-plane capacity

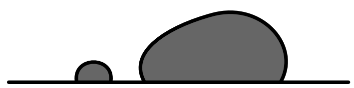


Fig. 4.1 A hull K is the shaded area in the picture. The set K is a compact subset of the closed upper half-plane $\overline{\mathbb{H}}$ such that its complement $\mathbb{H} \setminus K$ is simply connected. In particular, the set $K \cup \mathbb{R}$ is connected.

We start by a definition of a basic object of study. We are going to consider conformal maps from subsets of the upper half-plane onto the upper half-plane. By the Riemann mapping theorem the subset needs to be simply connected. Therefore we make the following definition.

Definition 4.1. A set $K \subset \overline{\mathbb{H}}$ is called a *hull* if K is compact and $\mathbb{H} \setminus K$ is simply connected.

Remark 4.1. An alternative definition of a hull as a subset of the open upper half-plane can be made in the following way: $K \subset \mathbb{H}$ is a hull if K is bounded, relatively closed in $\mathbb{H}$ and $\mathbb{H} \setminus K$ is simply connected. The two definitions are practically the same. To move from the former to the latter, one needs just to take the intersection of the set with the open upper half-plane and to move to the other direction, one needs to take the closure of the set.

Lemma 4.1. *For any hull K, there exists a unique conformal and onto map g_K : $\mathbb{H} \setminus K \to \mathbb{H}$ such that*

$$\lim_{z \to \infty} (g_K(z) - z) = 0 \tag{4.1}$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ *such that* $|z_n| \to \infty$. *Such* g_K *is said to have* hydrodynamic normalization. *Near* ∞, g_K *has the expansion*

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots$$

where the coefficients a_k, $k \in \mathbb{N}$, *are real.*

Proof. If $\tilde{g} : \mathbb{H} \setminus K \to \mathbb{D}$ is a conformal onto map, then $\tilde{g}(\infty) \in \partial\mathbb{D}$ is well-defined since there is a holomorphic extension of $z \mapsto \tilde{g}(-1/z)$ to a neighborhood of 0 by Theorem 3.3. Hence we can compose $\tilde{g}$ with a Möbius map from $\mathbb{D}$ onto $\mathbb{H}$ mapping $\tilde{g}(\infty)$ to ∞ and get this way a conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ mapping ∞ to ∞. By this observation and by the Riemann mapping theorem, there are conformal onto maps from $H = \mathbb{H} \setminus K$ onto $\mathbb{H}$ which map ∞ to ∞. Pick one of them and call it $\hat{g}$. Let $H' = \{-1/z : z \in H\}$ and

$$f(z) = -1/\hat{g}(-1/z). \tag{4.2}$$

By Theorem 3.3, f extends holomorphically and injectively to a neighborhood of 0. Let $\varepsilon > 0$ be such that $B(0,\varepsilon) \cap \mathbb{H} \subset H'$. Then f maps $(-\varepsilon, \varepsilon)$ into $\mathbb{R}$. Moreover, if $f = u + iv$, then $f'(0) = \partial_x u(0) = \partial_y v(0) > 0$ because f maps $B(0,\varepsilon) \cap \mathbb{H}$ into $\mathbb{H}$. Hence

$$f(z) = b_1 z + b_2 z^2 + \dots$$

near 0 where the coefficients satisfy $b_1 > 0$ and $b_j \in \mathbb{R}$. For $\hat{g}$ this implies that for large $|z|$

$$\hat{g}(z) = \hat{a}_{-1} z + \hat{a}_0 + \hat{a}_1 z^{-1} + \hat{a}_2 z^{-2} + \dots \tag{4.3}$$

where the coefficients satisfy $\hat{a}_{-1} > 0$ and $\hat{a}_j \in \mathbb{R}$. Now we notice that $\hat{g}$ satisfies (4.5), if and only if $\hat{a}_{-1} = 1$ and $\hat{a}_0 = 0$.

By the remark after the Riemann mapping theorem (Theorem 3.1), if $g : H \mapsto \mathbb{H}$ is a conformal onto map taking ∞ to ∞, then all the other such maps can be written as $\phi \circ g$ where ϕ is a Möbius self-map of $\mathbb{H}$ fixing ∞. The Möbius self-maps of $\mathbb{H}$ that fix ∞ are of the form

$$z \mapsto \alpha z + \beta$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. Hence for given $\hat{g}$ there is a unique choice for ϕ such that $g_K = \phi \circ \hat{g}$ has the expansion

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \dots$$

for $z \in \mathbb{H} \setminus B(0,R)$. □

Lemma 4.2. *The coefficient* a_1 *is non-negative and* $a_1 = 0$ *only if* g_K *is an identity map.*

Proof. Define a harmonic function h in $\mathbb{H} \setminus K$ by

$$h(z) = \operatorname{Im}(z - g_K(z)).$$

Then the boundary values of h are non-negative: it is zero on $\mathbb{R}$ away from K and on $\partial K \cap \mathbb{H}$ it is positive. Also $h(z) \to 0$ as $|z| \to \infty$. Hence by the minimum principle,

h is non-negative in $\mathbb{H}\setminus K$. In fact, h is strictly positive unless $h=0$ identically and g_K is an identity map. Now

$$\lim_{y\nearrow\infty} y\,h(\mathrm{i}y) = \lim_{y\nearrow\infty} y\,\mathrm{Im}\left(-\frac{a_1}{\mathrm{i}y} + \mathscr{O}\left(|y|^{-2}\right)\right) = a_1$$

which shows that $a_1 \geq 0$. The strict positivity follows when we notice that

$$a_1 = \frac{2R}{\pi}\int_0^{\pi} h(Re^{i\theta})\sin\theta\,\mathrm{d}\theta. \tag{4.4}$$

That formula follows from the previous formula and from the solution of the Dirichlet problem

$$\begin{cases} \Delta u &= 0 & \text{in } \{z\in\mathbb{H} : |z|>R\} \\ u(x) &= 0 & \text{for } x\in\mathbb{R}, |x|\geq R \\ u(Re^{i\theta}) &= \phi(\theta) & \text{for } \theta\in(0,\pi) \end{cases}$$

in terms of a Poisson kernel. The proof of the formula (4.4) is left as an exercise. □

Definition 4.2. If K is a hull and g_K satisfies the hydrodynamic normalization, then the coefficient of z^{-1} in the expansion of g_K is denoted by $a_1(K)$. We call $a_1(K)$ as the *half-plane capacity* of K.

The half-plane capacity satisfies the following properties:

- Scaling rule: $a_1(\lambda K) = \lambda^2 a_1(K)$ because

$$g_{\lambda K}(z) = \lambda g_K(\lambda^{-1}z) = z + \lambda^2 a_1(K) z^{-1} + \ldots$$

- Summation rule: $a_1(K\cup L) = a_1(K) + a_1(g_K(L))$. Let $L' = g_K(L)$. Then

$$g_{K\cup L}(z) = g_{L'}\circ g_K(z) = z + (a_1(K) + a_1(L'))z^{-1} + \ldots$$

- Translation invariance: $a_1(K+x) = a_1(K)$

$$g_{K+x}(z) = x + g_K(z-x) = z + a_1(K)z^{-1} + \ldots$$

From the summation rule and from Lemma 4.2 it follows that if $J\subset K$ are hulls then $a_1(J)\leq a_1(K)$ and $a_1(J) = a_1(K)$ only if $\mathbb{H}\cap(K\setminus J)=\emptyset$. We say that *half-plane capacity is increasing*. These properties make the half-plane capacity very natural measure for the size of the hull K (as seen from the point ∞ in the domain $\mathbb{H}$).

Example 4.1 (Half-disc). When $K=\overline{\mathbb{H}\cap B(x_0,R)}$, the corresponding map is

$$g_K(z) = z + \frac{R^2}{z-x_0} = z + \frac{R^2}{z} + \frac{R^2 x_0}{z^2} + \ldots$$

This expression can be verified by a direct computation: namely $g_K(x)\in\mathbb{R}$ when $x\in\mathbb{R}$, $|x-x_0|\geq R$, and for $\theta\in(0,\pi)$,

$$g_K(x_0 + Re^{i\theta}) = x_0 + 2R\cos\theta \in \mathbb{R}.$$

The half-plane capacity of the half-disc of radius R is $a_1(K) = R^2$.

Example 4.2 (Vertical line segment). When $K = [x_0, x_0 + ih] = \{x_0 + iy : y \in [0,h]\}$

$$g_K(z) = x_0 + \sqrt{(z-x_0)^2 + h^2} = x_0 + z\sqrt{1 - \frac{2x_0}{z} + \frac{x_0^2 + h^2}{z^2}}$$
$$= x_0 + z\left(1 - \frac{x_0}{z} + \frac{x_0^2 + h^2}{2z^2} - \frac{1}{8}\frac{4x_0^2}{z^2} + \dots\right) = z + \frac{h^2}{2z} + \dots$$

where we used the expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$ The half-plane capacity of the vertical line segment of length h is $a_1(K) = h^2/2$.

The following result on the inverse maps f_K of g_K can be shown, for instance, by imitating the proof of Lemma 4.1. The last claim follows from composing $g_K \circ f_K$ which is a conformal self-map in the upper half-plane which has the expansion $z + (a_1(K) + b_1(K))z^{-1} + \mathcal{O}(|z|^{-2}) = z + \mathcal{O}(|z|^{-1})$ as $z \to \infty$. Consequently, $g_K \circ f_K$ is the identity map and $a_1(K) + b_1(K) = 0$.

Lemma 4.3. *For any hull K, there exists a unique conformal and onto map $f_K : \mathbb{H} \to \mathbb{H} \setminus K$ such that*

$$\lim_{z\to\infty}(f_K(z) - z) = 0 \tag{4.5}$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ such that $|z_n| \to \infty$. Such f_K is also said to have hydrodynamic normalization. *Near ∞, f_K has the expansion*

$$f_K(z) = z + b_1 z^{-1} + b_2 z^{-2} + \dots$$

where the coefficients b_k, $k \in \mathbb{N}$, are real. Furthermore, $f_K = g_K^{-1}$ and $b_1(K) = -a_1(K)$.

4.1.1.1 Continuity of the half-plane capacity

We conclude this section by showing that the half-plane capacity is a continuous function of the hull. For a hull K and $\varepsilon > 0$, let K^ε be the *ε-thickening* of K, that is, K^ε is the smallest hull containing the set $\mathbb{H} \cap \bigcup_{z\in K} \overline{B(z,\varepsilon)}$.

Lemma 4.4. *There are constants $C(R) > 0$ and $\alpha > 0$ such that the following holds: If $K \subset K^\varepsilon \subset B(z_0, R)$, then*

$$a_1(K) \le a_1(K^\varepsilon) \le a_1(K) + C(R)\varepsilon^\alpha$$

Proof. The inequality on the left follows from the summation rule and positivity of the half-plane capacity.

To show the other inequality consider the harmonic functions $h_K(z) = \mathrm{Im}(z - g_K(z))$ and $h_{K^\varepsilon}(z) = \mathrm{Im}(z - g_{K^\varepsilon}(z))$. Note that they are both non-negative and bounded by R and note also that they are continuous in $\overline{\mathbb{H}\setminus K}$ and $\overline{\mathbb{H}\setminus K^\varepsilon}$, respectively.

Let $z \in \mathbb{H} \cap \partial K^\varepsilon$. Then $\mathrm{dist}(z,K) = \varepsilon$. Let P^z be the law of a complex Brownian motion send from z and let τ be the hitting time of $\mathbb{R} \cup K$. Then by Lemma 3.1

$$h_K(z) = \mathsf{E}^z[\mathrm{Im}\, B_\tau]$$

and by definition $h_{K^\varepsilon}(z) = \mathrm{Im}\, z$. Write

$$\begin{aligned} |h_K(z) - h_{K^\varepsilon}(z)| &\le \mathsf{E}^z[|\,\mathrm{Im}\, B_\tau - \mathrm{Im}\, z|] \\ &= \mathsf{E}^z[|\,\mathrm{Im}\, B_\tau - \mathrm{Im}\, z|\,;\, \sigma < \tau] + \mathsf{E}^z[|\,\mathrm{Im}\, B_\tau - \mathrm{Im}\, z|\,;\, \sigma = \tau] \end{aligned} \quad (4.6)$$

where is σ be the exit time from $(\mathbb{H}\setminus K) \cap B(z,\sqrt{\varepsilon})$. The first term on the right of (4.6) is at most $R\,\mathsf{P}^z[\sigma < \tau]$ and hence by Lemma 4.5 below, there are constants $\tilde{\alpha} > 0$ and $\tilde{C} > 0$ such that the first term is bounded by $\tilde{C}R(\varepsilon/\sqrt{\varepsilon})^{\tilde{\alpha}} = \tilde{C}R\varepsilon^{\tilde{\alpha}/2}$. The second term is at most $\sqrt{\varepsilon}$.

Now since for some constants $C(R) > 0$ and α, $|h_K(z) - h_{K^\varepsilon}(z)| \le C(R)\varepsilon^\alpha$ on the boundary of $\mathbb{H}\setminus K^\varepsilon$ and $h_K - h_{K^\varepsilon}$ is a bounded harmonic function on $\mathbb{H}\setminus K^\varepsilon$, the maximum principle gives that $|h_K(z) - h_{K^\varepsilon}(z)| \le C(R)\varepsilon^\alpha$ on $\mathbb{H}\setminus K^\varepsilon$. Therefore the formula (4.4) can be applied to show that $|a_1(K) - a_1(K^\varepsilon)| \le C(R)\varepsilon^\alpha$. □

Lemma 4.5 (Weak Beurling estimate for Brownian motion). *There exist constants $\alpha > 0$ and $C > 0$ such that the following holds: Let $U \subset \mathbb{D}$ be a domain such that $\mathbb{D}\setminus U$ contains a connected set containing 0 and a point in $\mathbb{C}\setminus\mathbb{D}$. Let P^z be the law of complex Brownian motion $(B_t)_{t\in\mathbb{R}_{\ge 0}}$ send from $z \in U$ and let τ be its exit time from U. Then for any $z \in U$*

$$\mathsf{P}^z[|B_\tau| = 1] \le C|z|^\alpha$$

Remark 4.2. The result is called weak since the proof below only gives that there is some exponent $\alpha > 0$. It doesn't give the optimal exponent which is $\alpha = 1/2$.

Proof. Consider a complex brownian motion send from $w \in \mathbb{C}$ with $|w| = 2$ and let $\sigma = \inf\{t \in \mathbb{R}_+ : |B_t| = 1 \text{ or } 4\}$. By rotational invariance of the complex Brownian motion

$$q = \mathsf{P}^w[B([0,\tau]) \text{ contains a loop around } 0]$$

is independent of w. Now $q > 0$ follows from a more general fact that the probability that d-dimensional Brownian motion follows any given continuous path with a given precision up to a given time is positive.

Let $\rho = |z|$. Now if $|B_\tau| = 1$ then the Brownian motion $B_t, 0 \le t \le \tau$, will hit the circles of radii $r_k = \rho\, 2^k$, $k = 0,1,2,\ldots,n_0(\rho)$ centered at 0 where $n_0(\rho)$ is the largest integer n such that $\rho\, 2^n \le 1$. Denote the hitting times of those circles by T_k, $k = 0,1,\ldots,n_0(\rho)$. If for some $k = 0,1,\ldots,n_0(\rho)-1$, B_t, $t \ge T_k$, makes a loop around 0 before hitting the circles of radii r_{k-1} or r_{k+1}, then the Brownian motion hits ∂U and $|B_\tau| < 1$. Apply the (strong) Markov property of Brownian motion for T_k, $k = 0, 1, \ldots, n_(\rho) - 1$, to show that

$$\mathsf{P}^z[|B_\tau| = 1] \leq (1-q)^{n_0(\rho)}.$$

Then note that $n_0(\rho) > (\log(1/\rho))/(\log 2) - 1$ and hence the claim holds for $C = 1/(1-q)$ and $\alpha = (\log 1/(1-q))/(\log 2)$. □

4.1.2 Growing families of hulls

Let I be an interval of the form $[0,\infty), [0,T]$ or $[0,T)$ where $T \in (0,\infty)$. Let $\gamma : I \to \overline{\mathbb{H}}$ be curve such that $\gamma(0) \in \mathbb{R}$. We can define a family of hulls $(K_t)_{t\in I}$ associated to $\gamma(t)$, $t \in I$, in the following way:

- If γ is simple (a curve is simple if and only if it is injective) and $\gamma(t) \subset \mathbb{H}$, $t > 0$, then define $K_t = \gamma([0,t])$ for any $t \in I$.
- If γ is not simple let H_t be the unbounded connected component of $\mathbb{H} \setminus \gamma([0,t])$ and let $K_t = \overline{\mathbb{H} \setminus H_t}$.

If γ is simple both of the above definition would give the same hulls $(K_t)_{t\in I}$.

Let $(K_t)_{t\in I}$ be a family of hulls parametrized by a real variable $t \in I$ where I is as above. The family of hulls associated to a curve is a good example of such family. If the family $(K_t)_{t\in I}$ is *growing* in the sense that $K_s \subset K_t$ for $s \leq t$ and if *the growth is continuous* in the sense that for any $\varepsilon > 0$ and for any $S \in (0,\infty)$ such that $[0,S] \subset I$ there exist $\delta > 0$ such that $K_{t+\delta} \subset K_t^\varepsilon$ for any $0 \leq t \leq S - \delta$, then by Lemmas 4.2 and 4.4, the function $\phi : t \mapsto a_1(K_t)$ is continuous and non-decreasing. If we assume that $K_0 \subset \mathbb{R}$ and that $\mathbb{H} \cap (K_t \setminus K_s) \neq \emptyset$ for any $0 \leq s < t \leq T$, then $\phi(0) = 0$ and by the summation rule and by the positivity of the half-plane capacity $\phi(t) > \phi(s)$ for any $0 \leq s < t \leq T$. Hence we can reparametrize the family of hulls by setting $\tilde{K}_t = K_{\phi^{-1}(2t)}$. In this parametrization $a_1(\tilde{K}_t) = \phi(\phi^{-1}(2t)) = 2t$. This can be summarized by saying that continuously growing families of hulls can be parametrized by capacity.

Definition 4.3. A family of hulls $(K_t)_{t\in[0,T)}$ is said to be *parametrized with the half-plane capacity* if $a_1(K_t) = 2t$. A curve $\gamma : [0,T) \to \overline{\mathbb{H}}$ is said to be *parametrized with the half-plane capacity* if the associated hulls are parametrized with the half-plane capacity.

For a given family of hulls $(K_t)_{t\in I}$ it is convenient to set

$$g_t = g_{K_t}.$$

If $(K_t)_{t\in[0,T)}$ is parametrized by the capacity then

$$g_t(z) = z + \frac{2t}{z} + \ldots$$

From now on we assume (almost without exceptions) that g_t is a conformal map with this form of an expansion near ∞. Often it is useful to call the parameter t as time.

Remark 4.3. The factor 2 is because of historical reasons: using that normalization the Loewner equation in $\mathbb{H}$ will be better compatible with the Loewner equation in $\mathbb{D}$. The choice, that the half-plane capacity is linear in t, is consistent with the summation rule of the half-plane capacity.

4.2 Loewner chains

4.2.1 Loewner equation holds for simple curves

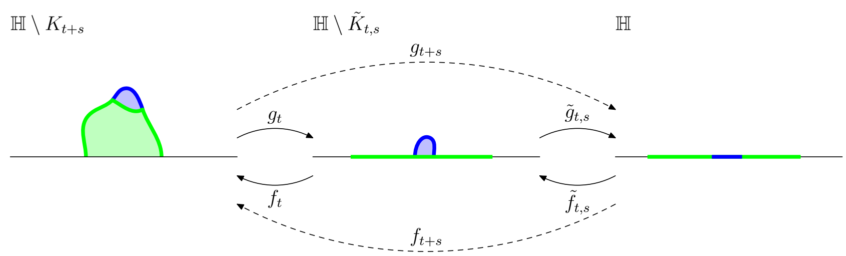


Fig. 4.2 Composition of hydrodynamical conformal maps uses the uniqueness of such maps. Therefore for instance $g_{t+s}(z) = \tilde{g}_{t,s} \circ g_t(z)$ for $z \in \mathbb{H} \setminus K_{t+s}$ and $f_{t+s}(z) = f_t \circ \tilde{f}_{t,s}(z)$ for $z \in \mathbb{H}$.

Let $(K_t)_{t \in [0,T]}$ be a growing family of hulls parametrized with the half-plane capacity. The Loewner differential equation describes infinitesimal changes in the conformal maps g_t as t increases. In its simplest version, the growth need to be *local* and consequently the Loewner equation contains one driving term. We will first look at this statement somewhat heuristically and then formulate and prove the result that the conformal maps g_t associated to a simple curve γ satisfy the Loewner equation.

We can write g_{t+s} as a composition of conformal maps in the following way, see also Figure 4.2. Let for t,s such that $t, t+s \in [0,T]$,

$$\tilde{K}_{t,s} = \overline{g_t(K_{t+s} \setminus K_t)}, \qquad \tilde{g}_{t,s} = g_{\tilde{K}_{t,s}}.$$

Notice that $a_1(\tilde{K}_{t,s}) = 2s$ by additivity of the half-plane capacity and that $g_{t+s} = \tilde{g}_{t,s} \circ g_t$ by uniqueness of the hydrodynamically normalized conformal maps.

Next we observe that we can apply the Poisson kernel of the upper half-plane to the inverses of intermediate conformal maps $\tilde{g}_{t,s}$. For any f_K, the function $h(z) = \mathrm{Im}(f_K(z) - z)$ is harmonic in $\mathbb{H}$ and bounded and non-negative in $\overline{\mathbb{H}}$. Consequently, by taking (3.3) and its harmonic conjugate, it follows that

$$f_K(z) - z = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} f_K(\xi)}{z - \xi} \, \mathrm{d}\xi. \tag{4.7}$$

Define $\tilde{f}_{t,s} = \tilde{g}_{t,s}^{-1}$. Then it holds that

$$\tilde{f}_{t,s} = g_t \circ f_{t+s}.$$

Let $t \geq 0$ and $\delta > 0$. We write

$$g_{t+\delta}(z) - g_t(z) = g_{t+\delta}(z) - \tilde{f}_{t,\delta} \circ g_{t+\delta}(z) \tag{4.8}$$

We say that the growth is local, when the support $\{\xi \in \mathbb{R} : \operatorname{Im} \tilde{f}_{t,\delta}(\xi) > 0\}$, tends to a point, which we denote by $W(t)$, as $\delta \to 0$ (we will give more precise definition in the next subsection). Then it follows (4.7) and (4.8) under suitable conditions that g_t satisfies the *Loewner differential equation* (of the upper half-plane)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}.$$

Here the number 2 in the numerator follows from the choice of the parametrization so that $a_1(\tilde{K}_{t,\delta}) = 2\delta$. We interpret that K_t is growing *locally at the point* $P(t) = g_t^{-1}(W(t))$ if g_t^{-1} extends continuously to $W(t)$.

The following theorem makes the above argument more formal.

Theorem 4.1. *Let $T > 0$ and let $\gamma : [0,T] \to \mathbb{C}$ be a simple curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(\,(0,T]\,) \subset \mathbb{H}$. Suppose that γ is parametrized by the capacity. Then*

$$W(t) = \lim_{z \to \gamma(t)} g_t(z) \tag{4.9}$$

exists for any $t \in [0,T]$ and $t \mapsto W(t)$ is continuous. Here the limit is along any sequence $z_n \in \mathbb{H} \setminus \gamma(0,t]$ converging to $\gamma(t)$. Moreover the hydrodynamically normalized conformal maps $(g_t)_{t \in [0,T]}$ related to γ satisfy the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{4.10}$$

with the initial value $g_0(z) = z$.

Before the proof of this theorem we present three auxiliary results.

Lemma 4.6. *Let K be a hull and $H = \mathbb{H} \setminus K$. If $K \subset B(x_0, r)$, then g_K maps $H \cap B(x_0, 2r)$ into $B(x_0, 3r)$ and $\sup_{z \in H} |g_K(z) - z| \leq 5r$.*

Proof. We can assume that $x_0 = 0$. Otherwise consider the map $g_{K - x_0}(z) = g_K(z + x_0) - x_0$.

Let $\tilde{g}$ be the holomorphic extension of $r^{-1} g_K(rz)$ to $\mathbb{D}^*$. Then $\tilde{g} \in \Sigma$ and by the Area theorem $\sum_{n=1}^{\infty} n |a_n(K)|^2 r^{-2(n+1)} \leq 1$ and therefore $|a_n(K)| \leq r^{n+1}$. Hence

$$|g_K(z) - z| \le \sum_{n=1}^{\infty} |a_n(K)|\,|z|^{-n} \le r \sum_{n=1}^{\infty} (r/|z|)^n = \frac{r^2}{|z| - r} \le r$$

for $|z| \ge 2r$. And therefore $g_K(\mathbb{H} \cap B(0,2r)) \subset B(0,3r)$.

If $z \in H \cap B(0,2r)$, then $|g_K(z) - z| \le |g_K(z)| + |z| < 5r$. □

Using the next lemma we can control the length distortion under conformal maps. This lemma could be used in the proof of the general result Theorem 3.4 about the continuity of conformal maps to the boundary. The same principle of proof when systematized gives estimates for so called *extremal length*.

Lemma 4.7. *Let ϕ be a conformal map from open set $U \subset \mathbb{C}$ into $B(0,R)$. Let $z_0 \in \mathbb{C}$ and let $C(r) = U \cap \{z : |z - z_0| = r\}$ for any $r > 0$. Then*

$$\inf_{\rho < r < \sqrt{\rho}} \{\mathrm{Length}\,(\phi(C(r)))\} \le \frac{2\pi R}{\sqrt{\log 1/\rho}}. \tag{4.11}$$

Proof. Let $l(r) = \mathrm{Length}\,(\phi(C(r)))$. By the Cauchy–Schwarz inequality

$$\begin{aligned} l(r)^2 = \left(\int_{C(r)} |\phi'(z)|\,|\mathrm{d}z|\right)^2 &\le \int_{C(r)} |\mathrm{d}z| \int_{C(r)} |\phi'(z)|^2\,|\mathrm{d}z| \\ &\le 2\pi r \int_{z_0 + re^{i\theta} \in U} \left|\phi'\left(z_0 + re^{i\theta}\right)\right|^2 r\,\mathrm{d}\theta. \end{aligned}$$

Divide this by r and then integrate over r to find that

$$\int_0^{\infty} l(r)^2\, r^{-1}\,\mathrm{d}r \le 2\pi \int_U |\phi'(z)|^2\,\mathrm{d}x\mathrm{d}y = 2\pi \mathrm{Area}(\phi(U))$$

which implies that

$$\frac{1}{2}\log\frac{1}{\rho}\left(\inf_{\rho < r < \sqrt{\rho}} l(r)^2\right) \le \int_{\rho}^{\sqrt{\rho}} l(r)^2\, r^{-1}\,\mathrm{d}r \le 2\pi^2 R^2.$$

The claim follows by taking a square root. □

The next lemma is the application of the Poisson kernel to establish Theorem 4.1.

Lemma 4.8. *There exist an absolute constant $C > 0$ such that the following holds: If $K \subset B(x_0,r) \cap \mathbb{H}$ and $z \in \mathbb{H}$, $|z - x_0| \ge Cr$, then*

$$\left| f_K(z) - z + \frac{a_1(K)}{z - x_0} \right| \le \frac{Cra_1(K)}{|z - x_0|^2}$$

where $f_K = g_K^{-1}$

Proof. We can assume $x_0 = 0$. We can also assume that the boundary of K is a continuous. If not, then take a sequence K_n each having a continuous boundary and such that $f_{K_n} \to f_K$ uniformly in compact subsets of $\mathbb{H}$.

A similar argument as above shows that f_K has an expansion of the form (4.3) and a direct calculation then tells that

$$f_K(z) = z - a_1 z^{-1} + \dots$$

Let

$$h(z) = \operatorname{Im}(f_K(z) - z).$$

Then h is a bounded continuous function in $\overline{\mathbb{H}}$ and harmonic in $\mathbb{H}$. Hence we can write h using the Poisson kernel of $\mathbb{H}$ as

$$h(z) = \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} h(\xi)\, \mathrm{d}\xi.$$

We can use this formula to derive the harmonic conjugate of h. Notice that $h = \operatorname{Im} f_K$ on $\mathbb{R}$ and therefore

$$f_K(z) = z + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi. \tag{4.12}$$

The additive constant in the harmonic conjugate of h was fixed by the expansion near ∞.

Now clearly $\operatorname{Im} f_K(\xi)$ is zero outside a bounded interval I which is defined as the smallest interval containing $\{\xi \in \mathbb{R} : f_K(\xi) \in \mathbb{H} \cap \partial K\}$. From this it follows that

$$\begin{aligned} f_K(z) &= z + \frac{1}{\pi} \int_I \frac{1}{\xi - z} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi \\ &= z - \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_I \xi^{n-1} \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi \right) z^{-n} \end{aligned}$$

for large enough $|z|$. Hence

$$a_1 = \frac{1}{\pi} \int_I \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi$$

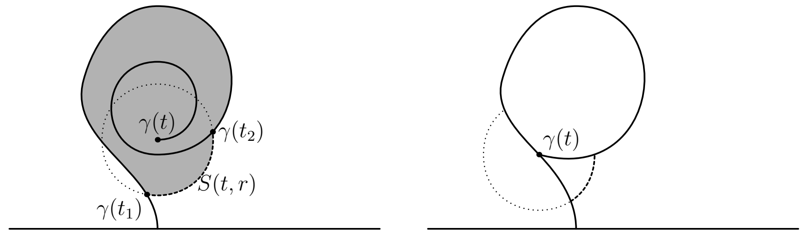
and

$$\begin{aligned} \left| f_K(z) - z + \frac{a_1(K)}{z} \right| &= \left| \frac{1}{\pi} \int_I \left(\frac{1}{\xi - z} + \frac{1}{z} \right) \operatorname{Im} f_K(\xi)\, \mathrm{d}\xi \right| \\ &\leq a_1(K) \sup \left\{ \left| \frac{1}{x - z} + \frac{1}{z} \right| : x \in I \right\} \end{aligned}$$

By Lemma 4.6, $I \subset (-3r, 3r)$ and hence

$$\left| \frac{x}{(x-z)z} \right| \leq \frac{6r}{|z|^2}$$

for any $|z| \geq 6r$ and $x \in I$. □



(a) The arc $S(t,r)$ of the circle of radius r centered at $\gamma(t)$ separates the tip $\gamma(t)$ from ∞ in H_t and it is outermost of all such arcs (in the sense that it separates all other such arcs from ∞ in H_t). When $\gamma(t) \in \mathbb{H}$ and r is small, the end points of $S(t,r)$ lie on the curve.

(b) When the curve is not simple, the conformal map doesn't extend continuously to the tip $\gamma(t)$ when that point is a double point visible from more than one side of the curve. The correct solution for this problem is to define the corresponding generalized boundary point as a nested sequence of arcs of circles. These generalized boundary points are called *prime ends*.

Fig. 4.3 Continuity of the conformal map g_t at the tip point $\gamma(t)$ follows from the fact that an arc $S(t,r)$ of a small circle is mapped to a set of small diameter.

Proof (Proof of Theorem 4.1). As usual, denote $H_t = \mathbb{H} \setminus \gamma(0,t]$. Since $\gamma[0,T]$ is bounded, we can define $R = \sup_{t\in[0,T]} |\gamma(t)| < \infty$.

For each $t \in [0,T]$ and $r > 0$, let $S(t,r)$ be the outermost of all the connected components of $H_t \cap \partial B(\gamma(t),r)$ which separate $\gamma(t)$ from ∞ in H_t. See Figure 4.3. Since by Lemma 4.6, g_t maps $H_t \cap B(0,2R)$ into $B(0,3R)$, we can apply Lemma 4.7 to g_t and show that the diameter of $g_t(S(t,r))$ is at most $6\pi R/\sqrt{\log(1/r)}$. Since the curves $g_t(S(t,r))$, $r > 0$ are disjoint and nested (in the sense that for any $0 < r_1 < r_2$, $g_t(S(t,r_2))$ separates $g_t(S(t,r_1))$ from ∞ in $\mathbb{H}$) and their diameters go to zero as $r \searrow 0$, there exist $W(t) \in \mathbb{R}$ such that

$$\{W(t)\} = \bigcap_{r>0} \overline{V(t,r)}$$

where $V(t,r)$ is the bounded component of $\mathbb{H} \setminus g_t(S(t,r))$.

Since γ is simple, $g_t(H_t \cap B(\gamma(t),r')) \subset V(r,t)$ for small enough $r' > 0$. Namely, when $r < \operatorname{Im} \gamma(t)$, the end points of $S(t,r)$ are points $\gamma(t_1), \gamma(t_2)$, for some $0 < t_1 \leq t_2 < t$. Since the distance from $\gamma(t)$ to $\gamma([t_1,t_2]) \cup S(t,r)$ is positive, then for small enough $r' > 0$, $S(t,r)$ separates $H_t \cap B(\gamma(t),r')$ from ∞ in H_t. See Figure 4.3(a). Therefore $g_t(H_t \cap B(\gamma(t),r')) \subset V(r,t)$ and

$$\{W(t)\} = \bigcap_{r'>0} \overline{g_t(H_t \cap B(\gamma(t),r'))}.$$

Hence $g_t(\gamma(t)) = \lim_{z\to\gamma(t)} g_t(z)$ is well-defined and the first claim follows.

Now for each $\varepsilon > 0$ there is $\delta > 0$ such that for any $t \in [0, T - \delta]$ we have that $g_t(\gamma(t,t+\delta]) \subset V(t,\varepsilon)$. Denote the conformal map associated to the hull

$g_t(\gamma([t,t+\delta]))$ by $\tilde{g}_{t,\delta}$ and let $\tilde{\gamma}(\delta) = g_t(\gamma(t+\delta))$. Since $\operatorname{diam} V(t,\varepsilon) \le r_0(\varepsilon) = 6\pi R/\sqrt{\log(1/\varepsilon)}$, $\tilde{\gamma}(\delta) \in \mathbb{H} \cap B(W(t), r_0(\varepsilon))$ and therefore by Lemma 4.6

$$|W(t+\delta) - W(t)| = |\tilde{g}_{t,\delta}(\tilde{\gamma}(\delta)) - W(t)| \le 3r_0(\varepsilon). \tag{4.13}$$

Therefore $t \mapsto W(t)$ is continuous.

Let $C > 0$ be as in Lemma 4.8. Let $t \in [0,T)$, $z \in H_t$ and choose $\varepsilon > 0$ so small that $(C+5)r_0(\varepsilon) < \operatorname{Im} g_t(z)$. If $0 < \delta \le T - t$ is such that $g_t(\gamma(t,t+\delta]) \subset V(t,\varepsilon)$ then

$$\begin{aligned} |g_{t+\delta}(z) - W(t)| &\ge |g_t(z) - W(t)| - |\tilde{g}_{t,\delta} \circ g_t(z) - g_t(z)| \\ &\ge C r_0(\varepsilon). \end{aligned}$$

Use Lemma 4.8 for the map $f_K = \tilde{g}_{t,\delta}^{-1}$ at point $g_{t+\delta}(z)$ with $r = r_0(\varepsilon)$ and $x_0 = W(t)$ to find that

$$\left| g_{t+\delta}(z) - g_t(z) - \frac{2\delta}{g_{t+\delta}(z) - W(t)} \right| \le \frac{2\delta C r_0(\varepsilon)}{|g_{t+\delta}(z) - W(t)|^2}.$$

Since we can take $r_0(\varepsilon) \searrow 0$ as $\delta \searrow 0$, the derivative from the right exists and satisfies

$$\partial_{t+} g_t(z) = \lim_{\delta \searrow 0} \frac{g_{t+\delta}(z) - g_t(z)}{\delta} = \frac{2}{g_t(z) - W(t)}$$

Since the right-hand side is continuous in t, actually, $\partial_t g_t(z)$ exists and we have shown that (4.10) holds. □

Example 4.3. Let $\delta(t) = 2\sqrt{t}$ and let $g_t(z) = \sqrt{z^2 + \delta(t)^2}$. Then

$$\partial_t g_t(z) = \frac{2}{g_t(z)}.$$

Therefore the driving term of the straight vertical line $t \mapsto i\delta(t)$, $t \ge 0$, is $W_t = 0$ for all t.

4.2.2 Solving Loewner equation with a continuous driving term

In this section, we study the solution of Loewner equation with a continuous driving term and show that there is a growing family of hulls parametrized with the half-plane capacity. In fact, we will show that there is one-to-one correspondence between *locally growing hulls* and the solutions of the Loewner equation with continuous driving terms.

Let $t \mapsto W_t$ be a given real valued function on $[0,T]$. We will investigate whether there is a family of conformal maps $(g_t)_{t \in [0,T]}$ that satisfy the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z. \tag{4.14}$$

4.2.2.1 The solution of Loewner equation is a normalized conformal map

First fix $z \in \overline{\mathbb{H}}$. Then (4.14) is just an ordinary differential equation (ODE)

$$\dot{z}_t = \frac{2}{z_t - W_t}, \qquad z_0 = z \tag{4.15}$$

in the parameter t.

Lemma 4.9. *For $z \in \overline{\mathbb{H}} \setminus \{W_0\}$, the solution of* (4.15) *is unique and exists for all $t \in [0,T] \cap [0, \tau(z))$ where $\tau(z) = \inf\{t \geq 0 : \liminf_{s \nearrow t} |z_s - W_s| = 0\}$. Furthermore, for fixed t, the map $z \mapsto z_t$ is continuous at any point $z \in \overline{\mathbb{H}} \setminus \{W_0\}$ such that $t < \tau(z)$.*

Proof. The mapping

$$\zeta \mapsto \frac{2}{\zeta - W_t} \tag{4.16}$$

is continuous in t and Lipschitz continuous in ζ in the set of points

$$\{(t,\zeta) \in [0,T] \times \overline{\mathbb{H}} : |\zeta - W_t| \geq \varepsilon\}$$

where $\varepsilon > 0$. Thus by the theory of ODEs, there exists a unique solution to (4.15) and the solution at given time is a continuous function of the initial condition. $\square$

Now set

$$g_t(z) = z_t$$

for $t \in [0,T] \cap [0, \tau(z))$ and $z \in \overline{\mathbb{H}} \setminus \{W_0\}$. We claim that this defines a conformal map. Define

$$H_t = \{z \in \mathbb{H} : \tau(z) > t\}.$$
$$K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}.$$

Then H_t is open by continuity of $z \mapsto g_t(z)$ and similarly K_t is closed.

Proposition 4.1. *The function g_t restricted to H_t is a conformal map onto $\mathbb{H}$. The set K_t is a hull.*

Proof. Let $z, z' \in \mathbb{H}$ and

$$D_t(z,z') = g_t(z) - g_t(z')$$

for any $t \in [0,T] \cap [0, \tau(z) \wedge \tau(z'))$. It satisfies the differential equation

$$\dot{D}_t(z,z') = -D_t(z,z') \, \frac{2}{(g_t(z) - W_t)(g_t(z') - W_t)}$$

which can be integrated as

$$D_t(z,z') = (z-z')\exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z)-W_s)(g_s(z')-W_s)}\right).$$

This show that g_t is one-to-one. Furthermore, the complex derivative $g_t'(z)$ exists and equals to

$$g_t'(z) = \lim_{z'\to z}\frac{D_t(z,z')}{z-z'} = \exp\left(-\int_0^t \frac{2\mathrm{d}s}{(g_s(z)-W_s)^2}\right).$$

This shows that g_t is holomorphic. Thus $g_t : H_t \to \mathbb{C}$ is a conformal map.

We will show that $g_t(H_t) = \mathbb{H}$. Note first, that

$$\partial_t \operatorname{Im} g_t(z) = -2\,\frac{\operatorname{Im} g_t(z)}{|g_t(z)-W_t|^2}$$

and hence $\operatorname{Im} g_t(z)$ is strictly decreasing and positive, since the previous formula can be integrated as

$$\operatorname{Im} g_t(z) = (\operatorname{Im} z)\,\exp\left(-\int_0^t \frac{2\mathrm{d}s}{|g_s(z)-W_s|^2}\right)$$

which holds for any $t \in [0,T]\cap[0,\tau(z))$. Therefore $g_t(H_t) \subset \mathbb{H}$. Next fix $t \in (0,T]$ and let $w \in \mathbb{H}$. Define $h_s(w)$ as the solution of the *backward Loewner equation*

$$\partial_s h_s(w) = -\frac{2}{h_s(w)-W_{t-s}}, \qquad h_0(w) = w. \tag{4.17}$$

Then $h_s(w)$, $0 \le s \le t$, is well-defined and lies in the upper half-plane, because $\operatorname{Im} h_s(w)$ is strictly increasing. Let $z = h_t(w)$. Then then $g_s(z) = h_{t-s}(w)$ because $s \mapsto h_{t-s}(w)$ solves the (forward) Loewner equation with the initial condition $h_t(w) = z$. In particular, it holds that $g_t(z) = w$ and we have shown that $g_t(H_t) = \mathbb{H}$.

Next we will show that K_t is a hull. Observe first that since g_t is conformal and $g_t(H_t) = \mathbb{H}$, $\mathbb{H}\setminus K_t = H_t$ is simply connected. As we stated above K_t is closed. To show that K_t is bounded let $M = \sup_{t\in[0,T]}|W_t|$. The first observation is that for any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z > M$, $\operatorname{Re} g_s(z)$ is strictly increasing since

$$\partial_s \operatorname{Re} g_s(z) = 2\frac{\operatorname{Re}(g_s(z))-W_s}{|g_s(z)-W_s|^2} > 0$$

when $\operatorname{Re} g_s(z) > M$. Similarly for any $z \in \overline{\mathbb{H}}$ with $\operatorname{Re} z < -M$, $\operatorname{Re} g_s(z)$ is strictly decreasing. The second observations is that for any $z \in \mathbb{H}$

$$\partial_s \operatorname{Im} g_s(z) = -2\frac{\operatorname{Im}(g_s(z))}{|g_s(z)-W_s|^2} \ge -\frac{2}{\operatorname{Im}(g_s(z))}$$

and hence when $\operatorname{Im} z > 2\sqrt{t}$,

$$(\operatorname{Im} g_t(z))^2 \geq (\operatorname{Im} z)^2 - 4t > 0.$$

Therefore

$$\left\{ z \in \mathbb{H} : |\operatorname{Re} z| > M \text{ or } \operatorname{Im} z > 2\sqrt{T} \right\} \subset H_t \tag{4.18}$$

and

$$K_t \subset \left\{ z \in \overline{\mathbb{H}} : |\operatorname{Re} z| \leq M \text{ and } \operatorname{Im} z \leq 2\sqrt{T} \right\}. \tag{4.19}$$

Now we have established that g_t is a conformal map from $H_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ and that K_t is a hull. Notice then that by using $(\operatorname{Re} g_t(z))^2 + M^2 \geq \max\{(\operatorname{Re} g_t(z))^2, M^2\} \geq (\operatorname{Re} z)^2$ and $(\operatorname{Im} g_t(z))^2 \geq (\operatorname{Im} z)^2 - 4t$, it follows that

$$g_t(z) = g_0(z) + \int_0^t \frac{2\mathrm{d}s}{g_s(z) - W_s} = z + \mathscr{O}(|z|^{-1})$$

as $z \to \infty$. We can apply Lemma 4.1 to show that g_t has the expansion

$$g_t(z) = z + \sum_{k=1}^{\infty} a_k(t) z^{-k}$$

which converges uniformly for $|z| > R$ where $R > 0$ satisfies $K_t \subset \overline{B(0,R) \cap \mathbb{H}}$. Hence $\partial_t g_t(z) = \frac{2}{z} + \ldots$ and consequently, $a_1(t) = 2t$. □

4.2.2.2 Local growth and Loewner chains

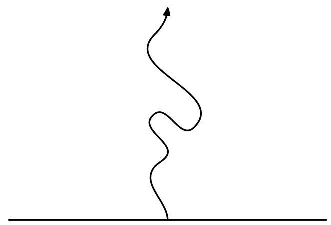
The following theorem will give a sufficient and necessary condition to the fact that g_t has a continuous driving term. This condition is called *local growth*. The result generalizes Theorem 4.1.

Theorem 4.2. *Let $(K_t)_{t \in [0,T]}$ be a growing family of hulls and g_t be the associated conformal maps. Then the following statements are equivalent:*

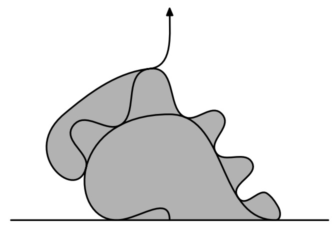
- *For all $t \in [0,T]$, $a_1(K_t) = 2t$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $t \in [0, T-\delta]$, there exists a bounded connected set $C \subset \mathbb{H} \setminus K_t$ with* $\operatorname{diam}(C) < \varepsilon$ *such that C separates $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.*
- *There is a continuous $W(t), t \in [0,T]$ such that g_t is the solution of* (4.14).

Definition 4.4. A *Loewner chain* is the solution g_t of the Loewner equation with a continuous driving term.

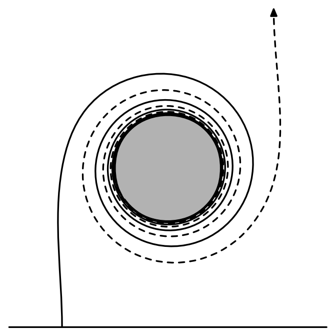
Remark 4.4. By the previous theorem, any one of the quantities $W(t), K_t, g_t$ could be taken as the most fundamental object. Hence the concept of Loewner chain includes all those features.



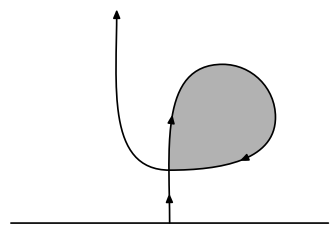
(a) A simple curve in $\mathbb{H}$ starting from the boundary



(b) A non-simple curve which doesn't have self-crossings and the increments $\gamma((t,t+s])$ are always visible to ∞ in $\mathbb{H} \setminus \gamma((0,t])$.



(c) A growing hull, which makes infinite number of turns around a disc



(d) A curve with a self-crossing

Fig. 4.4 Some examples and counterexamples based on Theorem 4.2: The growing hulls of (a)-(c) satisfy the "local growth" condition but (d) doesn't satisfy the condition. However it is possible to use the Loewner equation for (d), but then the driving term would have a discontinuity at the time of the self-crossing.

Example 4.4 (Some examples and counterexamples). See Figure 4.4 for some examples related to the theorem.

The curves in Figures 4.4(a) and 4.4(b) grow locally and hence they generate a Loewner chain. The former is a simple curve and already covered by Theorem 4.1. While the latter one isn't simple, it is non-self-crossing and thus satisfies the local growth conditions. We don't give the definition of non-self-crossing here formally, but the idea is explained in the figure.

Figure 4.4(c) indicates that the set of the locally growing hull collections is a strictly bigger class than the locally growing curves. In that example as the capacity time approaches t_0, the growing hull winds infinitely many times around a disc, and afterwards it unwinds infinitely many times. Hence it can't be continuous at t_0. For more information, see [5].

Finally Figure 4.4(d) shows a curve that violates the local growth. Any self-crossing leads to a jump in the Loewner driving term. Although we omit discussing it, considering Loewner driving terms with jumps is also very fruitful.

Proof (Proof of Theorem 4.2). The fact that the first statement implies the second one is a straightforward generalization of the proof of Theorem 4.1. Namely if $R > 0$ is such that $K_T \subset B(0,R)$ and $t, \varepsilon, \delta, C$ are as in the statement of the theorem, then $\mathrm{diam}(g_t(C)) \leq r_0(\varepsilon) = 6\pi R/\sqrt{\log 1/\varepsilon}$, because $C \subset \overline{B(z_0,\varepsilon)}$ for some $z_0 \in \mathbb{C}$ and by Lemmas 4.6 and 4.7 there is a circle of radius $\rho \in (\varepsilon, \sqrt{\varepsilon})$ which is mapped by g_t to a curve which has length less than $r_0(\varepsilon)$. Since $g_t(C)$ separates $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta} \setminus K_t)}$ from ∞ in $\mathbb{H}$, also the diameter of $\tilde{K}_{t,\delta}$ is less than $r_0(\varepsilon)$.

The intersection $\bigcap_{s>0} \tilde{K}_{t,s}$ is non-empty because the sets $\tilde{K}_{t,s}$ are compact and any finite intersection is non-empty. Since the diameter of $\bigcap_{s>0} \tilde{K}_{t,s}$ is less than $r_0(\varepsilon')$ for any $\varepsilon' > 0$, there exists $W(t) \in \mathbb{R}$ such that

$$\{W(t)\} = \bigcap_{s>0} \tilde{K}_{t,s}.$$

Now $\tilde{K}_{t,\delta} \subset B(W(t), r_0(\varepsilon))$ and therefore as in (4.13), the function $t \mapsto W(t)$ is continuous. The Loewner equation now holds by the same argument as in the end of the proof of Theorem 4.1. We have shown that the first statement implies the second one.

To prove that the second statement implies the first one, define for any $\delta > 0$, the oscillation of W by

$$O(W,\delta) = \sup\{|W(t) - W(s)| \, : \, s,t \in [0,T], |s-t| \leq \delta\}.$$

By continuity of W, $O(W,\delta) \searrow 0$ as $\delta \searrow 0$. Let $r_1(\delta) = \big((2\sqrt{\delta})^2 + O(W,\delta)^2\big)^{1/2}$. By the inclusion (4.18), $\tilde{K}_{t,\delta} = \overline{g_t(K_{t+\delta} \setminus K_t)} \subset B(W(t), r_1(\delta))$. By Lemmas 4.6 and 4.7 there exists an arc of a circle of radius $r \in (r_1(\delta), \sqrt{r_1(\delta)})$

$$S = \mathbb{H} \cap \partial B(W(t), r)$$

such that the length of $C = g_t^{-1}(S)$ is less than $cR/\sqrt{\log(1/r_1(\delta))}$, where $R > 0$ is such that $K_T \subset B(0,R)$ and $c > 0$ is some universal constant. Since S separates $\tilde{K}_{t,t+\delta}$ from ∞ in $\mathbb{H}$, C separates $K_{t+\delta} \setminus K_t$ from ∞ in H_t. Hence we have existence of the separating set C with a uniformly small diameter. The claim now follows. □

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