Chapter 3
Introduction to conformal mappings

In this chapter we present briefly some result of complex analysis which are useful for our theory. We review topics related to harmonicity, e.g. explicit formulas for Poisson kernels and relation to Brownian motion. We study continuity of conformal maps up to the boundary and review the distortion estimates.

We assume that the reader is familiar with Complex analysis on the level of Rudin's book [6]. This chapter is supplemented by Appendix C.

3.1 Harmonic functions

3.1.1 Mean value property and Poisson kernel

A *domain* is a non-empty, open and connected set. For a set A, $\overline{A}$ usually denotes its closure, whereas the meaning of A^* denotes on the context, but it is often related to the complex conjugation of to other reflections. Some domains that we will consider are the *unit disc* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the *exterior of the unit disc* $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ and the *upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Let U be a domain in the complex plane. A twice continuously differentiable function $u : U \to \mathbb{R}$ is *harmonic* if $\Delta u = 0$. A harmonic function $u : U \to \mathbb{R}$ has mean-value property in the sense that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\,\mathrm{d}\theta \tag{3.1}$$

for any $z \in U$ and $r > 0$ such that $\overline{B(0,r)} \subset U$. Conversely, if $u : U \to \mathbb{R}$ is continuous function that has the mean value property (3.1) for every $z \in U$ and for every $0 < r < r_0(z)$ (note that $r_0(z)$ can be strictly less than the distance to the boundary), then u is smooth and harmonic. See, for instance, [4] pp. 210, 218-220.

When the mean value property is applied together with a Möbius transformation,[1] the mean value property can be written for any point in the disc (not just for the center) as an integral over the boundary of the disc. Namely, if $u : \overline{B(0,R)} \to \mathbb{R}$ is continuous function that is harmonic in $B(0,R)$, then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|z - Re^{i\theta}|^2} u(Re^{i\theta}) \, \mathrm{d}\theta \tag{3.2}$$

where the quantity

$$P_{B(0,R)}(z,\theta) = \frac{R^2 - |z|^2}{|z - Re^{i\theta}|^2}$$

is called the *Poisson kernel* in $B(0,R)$. This extends to discs $B(z_0,R)$ in an obvious way by translation.

Similarly in the upper half-plane, if $u : \overline{\mathbb{H}} \to \mathbb{R}$ is continuous and bounded and if u is harmonic in $\mathbb{H}$ then u is given in terms of an integral of the *Poisson kernel* in $\mathbb{H}$ as

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} z}{|z - \xi|^2} u(\xi) \, \mathrm{d}\xi \tag{3.3}$$

for any $z \in \mathbb{H}$.

3.1.1.1 Harmonic conjugate

The *harmonic conjugate* of u is any harmonic function v such that $f = u + iv$ is holomorphic. If the function v exists, it is unique up to an additive constant. In a simply connected domains the harmonic conjugate exists. This can be seen from the Poisson kernel which can be written as

$$P_{B(0,R)}(z,\theta) = \operatorname{Re} \frac{Re^{i\theta} - z}{Re^{i\theta} + z}.$$

Therefore if we take the imaginary part of the complex valued kernel $(Re^{i\theta} - z)/(Re^{i\theta} + z)$, then the corresponding integral gives the harmonic conjugate in the disc. This can be summarized by an explicit formula for f in $B(0,R)$ given u

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z} u(Re^{i\theta}) \, \mathrm{d}\theta + iC$$

where $C \in \mathbb{R}$ is a constant.

In simply connected domains, the harmonic extension via chains of pair-wise overlapping discs is well-defined and hence v exists in the whole domain U. By considering $-if = v - iu$, we see that $-u$ is the harmonic conjugate of v.

[1] Harmonicity is preserved by any holomorphic change of coordinates.

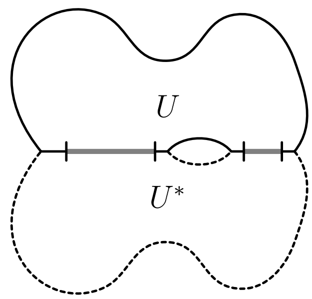


Fig. 3.1 By the Schwarz reflection principle, if U is a subdomain of the upper half-plane and a part of the boundary lies on the real axis and if the imaginary part of $f : U \to \mathbb{C}$ vanishes on $J \subset \mathbb{R}$, then f can be extended holomorphically to $U \cup J \cup U^*$

3.1.2 Schwarz reflection principle

Another consequence of the mean value property characterization of harmonic functions is the *Schwarz reflection principle*: if $f = u + iv$ is holomorphic in $D_+ = B(0,r) \cap \mathbb{H}$ and if $\lim_{z \to x} v(z) = 0$ as $z \in D_+$ tends to any $x \in (-r,r)$, then f has a unique holomorphic extension to $B(0,r)$. Namely, $v(\bar{z}) = -v(z)$ for any $z \in D_-$ defines a continuous extension of v to $B(0,r)$ and this extension satisfies the mean value property in $B(0,r)$. Hence v is smooth and harmonic in $B(0,r)$ and it has a harmonic conjugate which is unique if we require that $f = u + iv$ is in D_+. Hence f is well-defined and holomorphic in $B(0,r)$ and satisfies

$$f(\bar{z}) = \overline{f(z)}. \tag{3.4}$$

More generally, if $U \subset \mathbb{H}$ is a domain and $J \subset \mathbb{R} \cap \partial U$ is non-empty set such that each point $x \in J$ satisfies the condition that $B(x,r) \cap \mathbb{H} \subset U$ for some $r > 0$ and if $f : U \to \mathbb{C}$ is holomorphic function such that $\lim \operatorname{Im} f(z) = 0$ as z tends to J, then there exists a unique holomorphic extension of f to $U \cup J \cup U^*$ and the extension satisfies (3.4). Here U^* is the reflection of U with respect to the real axis.

3.1.3 Harmonicity and complex Brownian motion

Under suitable conditions on the domain U and on the function $h : \overline{U} \to \mathbb{R}$ harmonic in U and its boundary values $\phi = h|_{\partial U}$, the function h can be represented using the complex Brownian motion as

$$h(z) = \mathsf{E}^z[\phi(B_\tau)]$$

where τ is the exit time of $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ from U and E^z is the expected value with respect to the law of the complex Brownian motion $(B_t)_{t \in \mathbb{R}_{\geq 0}}$ sent from z.

Lemma 3.1. *Let U be a domain and $h : \overline{U} \to \mathbb{R}$ be a bounded continuous function such that h is harmonic in U. Let P^z be the law of a complex Brownian motion $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ started from $z \in U$ and E^z be the corresponding expected value. Assume that $\tau = \inf\{t \in \mathbb{R}_{\geq 0} : B_t \notin U\}$ is almost surely finite. Then $h(B_{t\wedge\tau})$ is a bounded continuous martingale and*

$$h(z) = \mathsf{E}^z \left[h\left(B_\tau\right)\right].$$

Proof. The fact that $M_t = h(B^{(z)}_{t\wedge\tau_z})$ is a local martingale follows from Itô's formula similarly as in the proof of the conformal invariance of Brownian motion. Since h is bounded, M_t is a bounded continuous martingale and we can use optional stopping to show that $M_0 = \mathsf{E}[M_{\tau_z}]$. □

3.2 Conformal maps

Definition 3.1. A map $f : U \to \mathbb{C}$ is a *conformal map* if and only if it is holomorphic and injective. A *univalent function* is the same as a conformal map.

When a map $f : U \to U'$ is *conformal and onto*, i.e., f is conformal and $f(U) = U'$, we state explicitly the fact that the map is onto.

If f is conformal, locally near z_0, we have the absolutely convergent expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots$$

It is necessary that $f'(z_0) \neq 0$ based on the expansion, otherwise f wouldn't be injective near z_0. Thus if we ignore the small correction of order $|z - z_0|^2$, locally the map f translates z_0 to $f(z_0)$, rotates around that point by multiplying by the complex number (of unit modulus) $f'(z_0)/|f'(z_0)|$ and scales by the factor $|f'(z_0)|$. The intuitive definition of a conformal map is that it is a map that is *locally a combination of translation, rotation and scaling*.

If $f : U \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$, then it is continuously invertible near z_0 and the inverse is holomorphic, [4] p. 165. Therefore the inverse of a conformal map is conformal.

However, the fact that $f' \neq 0$ everywhere is not sufficient for f to be injective globally. For example, consider the map $z \mapsto z^2$ in the domain $\mathbb{C} \setminus \{0\}$. Its derivative is non-zero everywhere, but it is not injective because $z^2 = (-z)^2$.

Example 3.1. The most elementary examples of conformal maps are the Möbius maps, which can be interpreted as the conformal self-maps of the *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. As linear fractional transformations they map any circle of the Riemann sphere onto a circle. Let's recall that the conformal self-maps of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and the conformal maps from the upper-half plane $\mathbb{H}$ onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are Möbius maps and of the form[2]

[2] In the first map, either a or b can be ∞. In that case, take the corresponding limit of the expression which is normalized in such a way that the limit is finite for fixed λ.

$$z \mapsto \lambda \frac{a-b}{a^2} \frac{z-a}{z-b}, \qquad z \mapsto \nu \frac{z-w}{z-\overline{w}},$$

respectively, where $\lambda > 0$, $a, b \in \mathbb{R}$ with $a \neq b$, $\nu \in \mathbb{T} = \partial\mathbb{D}$ and $w \in \mathbb{H}$. See also Appendix C.

3.2.1 Riemann mapping theorem

The Riemann mapping theorem establishes existence of conformal maps between simply connected domains.

A domain $U \subset \mathbb{C}$ is *simply connected* if its complement $\hat{\mathbb{C}} \setminus U$ in the Riemann sphere is connected. For example $S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$ is simply connected because the parts $\operatorname{Im} z \leq 0$ and $\operatorname{Im} z \geq 1$ can be connected through infinity. An equivalent definition of simply connectedness is that each closed loop in U is null-homotopic, that is, each loop can be continuously shrunk to a trivial loop. See [6] for more details.

Theorem 3.1 (Riemann mapping theorem). *Suppose $U \subset \mathbb{C}$ is a simply connected domain other than $\mathbb{C}$ and $w \in U$. Then there exist a unique conformal map f from U onto $\mathbb{D}$ such that $f(w) = 0$ and $f(w) > 0$.*

The proof can be found for instance from [6].

Remark 3.1. All the other conformal maps from U onto $\mathbb{D}$ are obtained by composing f with a Möbius self-map of the disc.

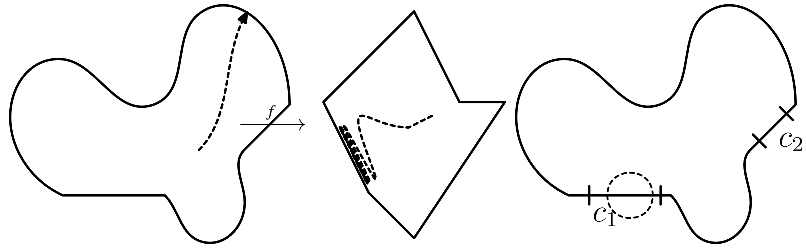
Remark 3.2. Notice that if $U \subset \hat{\mathbb{C}}$ is a simply connected domain and $w \notin U$, then the image $\tilde{U}$ of U under $z \mapsto 1/(z-w)$ is bounded. Therefore $\tilde{U}$ is a subset of $\mathbb{C}$. Consequently, by the Riemann mapping theorem, if $U_1, U_2 \subset \hat{\mathbb{C}}$ are simply connected domains and $\hat{\mathbb{C}} \setminus U_k$ contains at least two distinct points for $k = 1, 2$, then there exists a conformal map from U_1 onto U_2 and we say that U_1 and U_2 are conformally equivalent.

3.2.2 Continuity up to the boundary

In this section, we follow Ahlfors [1] and Pommerenke [5], see also [2].

Let's first see what type of continuity up to the boundary follows from the fact that ϕ is a homeomorphism, that is, a continuous map with a continuous inverse.

For that purpose, we define what we mean when we say that a sequence or a curve tends to the boundary domain. Let U be a non-empty open set, $z_n \in U$ a sequence and $\gamma : [0,1) \to U$ a curve. Remember that a *curve* in a topological space X is a continuous map from an interval of $\mathbb{R}$ into X. We say that (z_n) or $\gamma(t)$ *tends to the boundary* if (z_n) or $\gamma(t)$ will stay eventually away from any point in U, more



(a) If $f : U \to U'$ is a homeomorphism and $\gamma(t)$ is a curve that tends to the boundary, then the image $f(\gamma(t))$ tends to the boundary. However, it is not always true that $f(\gamma(t))$ extends continuously to its end point, not even when γ extends continuously to its end point.

(b) Schwarz reflection principle can be applied in those boundary arcs that are straight line segments and away from other parts of the boundary.

Fig. 3.2 By Theorems 3.2 and 3.3 conformal map maps boundary to boundary and extends continuously and injectively to a piece of boundary which is a straight line segment, an arc of a circle or an analytic curve.

formally, for each $z \in U$ there exist $\varepsilon(z) > 0$ and $n_0(z) \in \mathbb{N}$ such that $|z - z_n| \geq \varepsilon(z)$ for $n \geq n_0(z)$ or there exist $\varepsilon(z) > 0$ and $0 \leq t_0(z) < 1$ such that $|z - \gamma(t)| \geq \varepsilon(z)$ for $t_0(z) \leq t < 1$.

The discs $B(z, \varepsilon(z))$ form an open covering of U and for any compact $K \subset U$ there is a finite subcover. Hence we see that z_n or $\gamma(t)$ will stay eventually away from any compact $K \subset U$ in the sense that there exist $n_0(K) \in \mathbb{N}$ and $0 \leq t_0(K) < 1$ such that $z_n \notin K$ for $n \geq n_0(K)$ and $\gamma(t) \notin K$ for $t_0(K) \leq t < 1$. After noticing this the following theorem is almost trivial.

Theorem 3.2. *Let U and U' be non-empty open subsets of $\mathbb{C}$ and let $f : U \to U'$ be a homeomorphism. If (z_n) or $\gamma(t)$ tends to the boundary of U, then $(f(z_n))$ or $f(\gamma(t))$ tends to the boundary of U'.*

Proof. Let $K \subset U'$ be compact. Then by continuity of f^{-1}, the set $f^{-1}(K)$ is compact and there is $n_0 \in \mathbb{N}$ and $0 \leq t_0 < 1$ such that $z_n \notin f^{-1}(K)$ for $n \geq n_0$ and $\gamma(t) \notin f^{-1}(K)$ for $t_0 \leq t < 1$. Therefore $f(z_n) \notin K$ for $n \geq n_0$ and $f(\gamma(t)) \notin K$ for $t_0 \leq t < 1$. The claim follows by taking K to be a closed ball. □

Next we state and prove a theorem based on the Schwarz reflection principle that gives the continuity of f to the boundary arcs which are straight line segments.

Suppose that the boundary of U contains an open straight line segment c. By applying rotation and translation, we can assume that c is the interval $a < x < b$ on the real line. Suppose also that every point on c has an open neighborhood in $\mathbb{C}$ whose intersection with the whole boundary ∂U is the same as with the arc c. By this assumption each point in c is now a center of a disc whose diameter is a subset of c, and which c divides in to two half-discs which are either completely inside or outside of U. Notice that at least one of the half-discs is inside U. Since c

is connected, the property, whether one or two half-discs are inside U, is the same in each point. Therefore we can name these cases as *one-sided free arc* and *two-sided free arc*. See Figure 3.2(b) where c_1 and c_2 are one-sided free arcs.

Theorem 3.3 (Schwarz reflection principle for conformal maps). *Let U be a domain with one-sided free arc c. Then any conformal onto map $f : U \to \mathbb{D}$ can be extended to a holomorphic and injective map on $U \cup c$. The image of c is an arc c' on the unit circle $\partial\mathbb{D}$. Furthermore, if we apply the same extension to two or more one-sided free arcs, then the resulting extension is holomorphic and injective.*

Proof. Let c be one-sided free arc and $x \in c$ and D a half-disc neighborhood of x which is contained in U. We can assume that the point $f^{-1}(0)$ is not in D by choosing smaller D if necessary. Then $\log f(z)$ has single valued branch in D and its real part tends to 0 as $z \in D$ tends to c, because by the previous theorem $|f(z)|$ goes to 1. Therefore by the Schwarz reflection principle (3.4), $\log f(z)$ has holomorphic extension to $D \cup c \cup D*$ where D^* is the reflection of D with respect to $\mathbb{R}$. Therefore $f(z)$ can be extended holomorphically to a disc around z. The extensions in overlapping disc must coincide and therefore f has holomorphic extension to c and $|f(z)| = 1$ when $z \in c$. Call the neighborhood of c which lies outside U as U_-. Then f is now defined on $U \cup c \cup U_-$.

Clearly the extension is one-to-one if we manage to prove that $f(x) \neq f(x')$ for any $x, x' \in c$, $x \neq x'$ after all in $|f| < 1$ in U, $|f| = 1$ on c and $|f| > 1$ in U_- and in addition in U_-, f is by construction one-to-one. Assume that for some $x, x' \in c$, $x \neq x'$, $f(x) = f(x')$. We can assume that $f(x) = 1$.

Notice that $f'(x) \neq 0$ and $f'(x') \neq 0$. Otherwise $f(z) = c_0 + c_n(z-x)^n + \ldots$ around x, say, where $n \geq 2$ and $c_n \neq 0$. The interval $(1-\varepsilon, 1]$ would have n fold preimage under f and those paths would meet at angles $2\pi/n$ at x or x'. Since $n \geq 2$, at least one of them would intersect with D^* which leads to a contradiction. Thus $f'(x) \neq 0$ and $f'(x') \neq 0$ and f is locally holomorphically invertible near x and x'. A similar argument shows that any neighborhoods of x and x' are intersected by $f^{-1}(\{1-\varepsilon\})$ for small $\varepsilon > 0$. This leads to a contradiction with the injectivity of f in U. The last claim follows from the same argument. □

Remark 3.3. The previous theorem has a modification for c which is an arc of a circle or more generally for c which is an image of line segment under a holomorphic map (c is called an analytic arc).

A compact set $A \subset \mathbb{C}$ is said to be *locally connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any two points $a, b \in A$ with $|a-b| < \delta$, there exist a closed connected set B with $a, b \in B \subset A$ and $\operatorname{diam} B < \varepsilon$. For non-bounded closed $A \subset \hat{\mathbb{C}}$, we could adjust this definition and the next theorem by defining metric on the Riemann sphere $\hat{\mathbb{C}}$ that makes $\hat{\mathbb{C}}$ a compact space.

Theorem 3.4. *Let $U \subset \mathbb{C}$ be a bounded domain. A conformal onto map $f : \mathbb{D} \to U$ extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ if and only if ∂U is locally connected.*

For the proof, see Appendix C.

If $f : \mathbb{D} \to U$ is as in the previous theorem and if it extends continuously to the boundary, then ∂U is a closed curve that can be parametrized as $\theta \mapsto f(e^{i\theta})$. On the other hand, any closed curve is locally connected. Hence f extends continuously to the boundary if and only if the boundary is a curve. Clearly this extension is injective if and only if $\theta \mapsto f(e^{i\theta})$ is a simple curve. Hence the previous theorem implies that f extends to a continuous and injective map from $\overline{\mathbb{D}}$ onto $\overline{U}$ if and only if U is a Jordan domain.[3] In fact, the inverse map is in that case continuous to the boundary and any conformal map between two Jordan domains extends to a homeomorphism between their closures.

3.2.3 Schwarz–Christoffel maps

Conformal mappings that map the unit disc or the upper half-plane onto the interior of a polygon form a useful class of conformal mappings, because they have fairly explicit formulas. If a point p is mapped to a vertex of the polygon with interior angle $\alpha\pi$, the map looks locally like a constant times $(z-p)^{\alpha}$. The following theorem gives the precise statement.

Theorem 3.5. *Let U be the interior of a polygon γ with vertices $w_1, w_2, \dots, w_n$ and interior angles $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$. Then any conformal and onto map $f : \mathbb{H} \to U$ with $f(\infty) = w_n$ is of the form*

$$f(z) = C_1 + C_2 \int^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} \mathrm{d}\zeta \tag{3.5}$$

where C_1 and C_2 are constants and $w_k = f(z_k)$, $k = 1, 2, \dots, n-1$.

The proof is presented in Appendix C. The formula (3.5) is called the Schwarz-Christoffel formula.

Example 3.2. Let $n = 3$ and $\alpha_k = 1/3$ for all $k = 1, 2, 3$. Suppose that $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. Then any conformal map from $\mathbb{H}$ onto an equilateral triangle T such that z_1, z_2, z_3 are mapped to the vertices T, is of the form

$$f(z) = C_1 + C_2 \int^z \zeta^{-\frac{2}{3}} (\zeta - 1)^{-\frac{2}{3}} \mathrm{d}\zeta.$$

The pair of constants C_1, C_2 corresponds to the position, orientation and size of T.

Example 3.3. Let $n = 4$ and $\alpha_k = 1/2$ for all $k = 1, 2, 3, 4$. Suppose that $z_1 = 0$, $z_2 = x \in (0, 1)$, $z_3 = 1$ and $z_4 = \infty$. Then any conformal map from $\mathbb{H}$ onto a rectangle R such that z_1, z_2, z_3, z_4 are mapped to the vertices R, is of the form

$$f(z) = C_1 + C_2 \int^z \zeta^{-\frac{1}{2}} (\zeta - x)^{-\frac{1}{2}} (\zeta - 1)^{-\frac{1}{2}} \mathrm{d}\zeta.$$

[3] A curve is Jordan if it is simple closed curve. A domain is Jordan if it's boundary is Jordan curve.

The pair of constants C_1, C_2 still corresponds to the position, orientation and size of R. The value of $z_2 = x$ is treated as a parameter and it in one-to-one correspondence with the aspect ratio of R.

3.3 From Area theorem to distortion

In this section we present some classical result about the following two classes of functions:

Definition 3.2. The *class S* consists of all holomorphic and univalent functions in $\mathbb{D}$ such that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad |z| < 1. \tag{3.6}$$

The *class* Σ consists of all holomorphic and univalent functions in $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ such that

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots, \quad |z| > 1. \tag{3.7}$$

Notice that if $f \in S$, then

$$g(z) = 1/f(z^{-1}) = z - a_2 + (a_2^2 - a_3)z^{-1} + \dots \tag{3.8}$$

belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$. Conversely if g belongs to Σ and $g(z) \neq 0$ for all $z \in \mathbb{D}^*$, then

$$f(z) = 1/g(z^{-1}) = z - b_0 z^2 + (b_0^2 - b_1)z^3 + \dots$$

belongs to S.

The area of a bounded domain U, whose boundary is a smooth curve, can be computed as $\mathrm{Area}(U) = \frac{1}{2}\int_{\partial U} x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\int_{\partial U} \overline{w}\,\mathrm{d}w$. This is a consequence of so called Green's theorem which is a special case of Stokes' theorem for two dimensions. If $g \in \Sigma$ and we apply formula for the area inside $\theta \mapsto g(re^{i\theta})$, $r > 1$, we get the formula

$$\mathrm{Area}\left(\mathbb{C} \setminus g\left(\mathbb{D}^*\right)\right) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).$$

The reader can verify the details or see [3], p. 29, for the proof. The next theorem follows immediately from the area formula.

Theorem 3.6 (Area theorem). *For any $g \in \Sigma$,*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

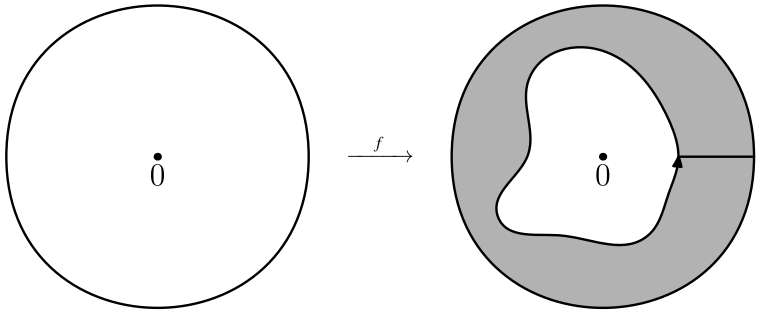


Fig. 3.3 A map f from $\mathbb{D}$ into $\mathbb{D}$ can be studied by the Loewner equation in $\mathbb{D}$ by defining a curve that first goes from $\partial\mathbb{D}$ to $\partial f(\mathbb{D})$ and then follows the boundary of the image domain $\partial f(\mathbb{D})$.

If $f \in S$ and the coefficients are as in (3.6), then there exists odd function[4] $h \in S$ such that $h(z) = \sqrt{f(z^2)}$ and

$$h(z) = z + \frac{1}{2}a_2 z^3 + \dots$$

The function h can be constructed as follows: The function $\phi(z) = \log(f(z)/z)$ has single-valued branch in $\mathbb{D}$, because $f(z)/z$ is holomorphic and doesn't have zeros in $\mathbb{D}$. Choose the branch for instance so that $\phi(0) = 0$. Hence $f(z) = z\exp\phi(z)$ and $h(z) = z\exp(\phi(z^2)/2)$ is in S and satisfies the required properties. Therefore (3.8) and the Area theorem imply that for any $f \in S$

$$|a_2| \leq 2. \tag{3.9}$$

The result (3.9) is called Bieberbach's theorem and it is a special case of the following famous and difficult theorem.

Theorem 3.7 (Bieberbach conjecture – de Branges theorem). *For any $f \in S$, $|a_n| \leq n$, $n = 2, 3, \dots$*

Remark 3.4 (A historical remark). In 1923, Charles Loewner (his birth name was Karel Löwner in Czech and he used also the name Karl Löwner as a German version of his name) was studying the Bieberbach conjecture in the paper where he introduced the Loewner equation. He was studying conformal maps from the unit-disc, and therefore he introduced the Loewner equation in $\mathbb{D}$ (for maps $f_t : \mathbb{D} \to D_t$) where it is written as

$$\partial_t f_t(z) = f_t'(z)\, z \frac{z + e^{iU_t}}{z - e^{iU_t}}$$

for a conformal map f_t from $\mathbb{D}$ onto a simply connected domain $D_t \subset \mathbb{D}$, $0 \in D_t$, normalized by the expansion near 0

[4] The function h is odd if $h(-z) = -h(z)$.

$$f_t(z) = e^{-t}z + \dots$$

The Loewner equation holds, for instance, when $D_t = \mathbb{D} \setminus \gamma((0,t])$ where $\gamma\colon [0,T] \to \mathbb{C}$ is a simple curve with $\gamma(0) \in \partial\mathbb{D}$ and $\gamma((0,T]) \subset \mathbb{D}$. The function $t \mapsto U_t$ is real and continuous.

Let $0 \in \hat{D} \subset \mathbb{D}$ be a simply connected domain. By approximation we can always assume that the boundary of $\hat{D}$ is a simple curve. By considering a curve $\gamma(t)$, $t \in [0,T]$, as in Figure 3.3 which first follows a curve from $\partial\mathbb{D}$ to $\partial\hat{D}$ (a line segment, say) and then follows $\partial\hat{D}$ in counterclockwise direction, say, we can use the Loewner equation to study the conformal map ϕ from $\mathbb{D}$ onto $\hat{D}$ satisfying $\phi(0) = 0$, $\phi'(0) > 0$, because $\phi = f_T$. Using this approach Charles Loewner was able to show that for any $f \in S$ (which has an expansion of the form (3.6))

$$|a_3| \leq 3$$

which is another special case of the Bieberbach–de Branges theorem.

3.3.1 Further consequences of Bieberbach's theorem

One of the consequences Bieberbach's theorem (3.9) is the following. Let's use $\operatorname{dist}(x,A)$ to denote the Euclidian distance from a point x to a set A.

Theorem 3.8 (Koebe $1/4$ theorem). *Let $f \in S$ and $U = f(\mathbb{D})$ then*

$$\frac{1}{4} \leq \operatorname{dist}(0,\partial U) \leq 1$$

Proof. Let $f \in S$ and $w \notin f(\mathbb{D})$. Suppose that the expansion of f is given by (3.6). Then $w \neq 0$ and the map

$$g(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \dots$$

is holomorphic and univalent in $\mathbb{D}$. The details are left to the reader. Thus by Bieberbach's theorem (3.9), $1/|w| - |a_2| \leq |1/w - a_2| \leq 2$ and consequently, $1/|w| \leq 4$, which gives the lower bound.

Let $d = \operatorname{dist}(0,\partial U)$. Define a conformal map h from $\mathbb{D}$ into $\mathbb{D}$ by $h(z) = f^{-1}(dz)$. Now $h'(0) = d/f'(0) = d$ and by the Schwarz lemma $|h'(0)| \leq 1$. □

To apply Bieberbach's theorem (3.9) to a less restricted class of functions, define for any f univalent in $\mathbb{D}$ and for any $w \in \mathbb{D}$ a function

$$h(z) = \frac{f\left(\frac{z+w}{1+\overline{w}z}\right) - f(w)}{(1-|w|^2)\,f'(w)} = z + \left(\frac{1}{2}(1-|w|^2)\frac{f''(w)}{f'(w)} - \overline{w}\right)z^2 + \dots$$

We leave as an exercise to verify the expansion. Since $h \in S$, this expansion and (3.9) imply the following result.

Proposition 3.1. *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ and if $z \in \mathbb{D}$ then*

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leq 4.$$

The previous result can be integrated (see [3], p. 32) to give Koebe's theorem, a result which, for example, tells how f distorts circles $\theta \mapsto re^{i\theta}$. The first of the inequalities tells that $\theta \mapsto f(re^{i\theta})$ lies between two particular circles centered at $f(0)$ and the second inequality tells that the length of this curve is bounded from below and from above by certain constants.

Theorem 3.9 (Koebe distortion theorem). *If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ and if $z \in \mathbb{D}$ then*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z)-f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}$$
$$|f'(0)|\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)|\frac{1+|z|}{(1-|z|)^3}$$

3.4 Harmonic measure

Recall the representation of harmonic functions using a Brownian motion of Section 3.1.3. Because such functions with piecewise constant boundary values are so important, we give such a function here a name.

Let U is a simply connected domain in $\mathbb{C}$ with a non-empty, locally connected boundary. Let $\phi : U \to \mathbb{D}$ is a conformal and onto map.

Definition 3.3. Let $z \in U$ and $E \subset \partial U$ a Borel set. Then *harmonic measure* of E relative to U seen from z is defined as

$$\mathrm{HM}(z,E,U) = \mathrm{HM}(w,\phi^{-1}(E),\mathbb{D}) = \frac{1}{2\pi}\int_{\phi^{-1}(E)} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,\mathrm{d}\theta$$

Remark 3.5. The function $z \mapsto \mathrm{HM}(z,E,U)$ is harmonic in U and tends to one as z tends to an interior (with respect to ∂U) point of E and to zero as z tends to an interior point of $\partial U \setminus E$. This observation leads to generalizations of the concept of harmonic measure to non-simply connected domains. For general domains, $\mathrm{HM}(z,E,U)$ can be defined as supremum over $h(z)$ where h is harmonic with continuous boundary values $f = h|_{\partial U}$ such that $f \leq 1$ on E and $f = 0$ on $\Omega \setminus E$.

Remark 3.6. If the boundary is non-simple and we wish to separete the "left-hand and right-hand sides" of boundary points point we can use the formula

$\mathrm{HM}(w, \phi^{-1}(E), \mathbb{D})$ to do so. In this approach, the points in $\partial\mathbb{D}$ parametrize $\partial\Omega$ and a generalized boundary point (also called prime end) is basically an equivalence class of convergent sequences z_n tending to the boundary of Ω such that $\phi(z_n)$ converges. Two sequences z_n and w_n are equivalent if $\phi(z_n)$ and $\phi(w_n)$ tend to the same limit point in $\partial\mathbb{D}$.

Lemma 3.2 (Weak Beurling estimate). *There exist constant $\alpha > 0$ and $C > 0$ such that the following holds: Let $D = \mathbb{D} \setminus \gamma[0,1)$ where $\gamma : [0,1) \to \mathbb{D}$ be a simple curve with $\gamma(0) = 0$ and $\lim_{t \nearrow 1} |\gamma(t)| = 1$. Let P^z be the law of complex Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ send from $z \in D$ and let τ be its exit time from D. Then for any $z \in D$*

$$\mathsf{P}^z[|B_\tau| = 1] \leq C|z|^\alpha$$

Remark 3.7. The result is called weak since the proof below only gives that there is some exponent $\alpha > 0$. It doesn't give the optimal exponent which is $\alpha = 1/2$.

Proof. Consider a complex brownian motion send from $w \in \mathbb{C}$ with $|w| = 2$ and let $\sigma = \inf\{t \in \mathbb{R}_+ : |B_t| = 1 \text{ or } 4\}$. By rotational invariance of the complex Brownian motion

$$q = \mathsf{P}^w[B([0,\tau]) \text{ contains a loop around } 0]$$

is independent of w. Now $q > 0$ follows from a more general fact that the probability that d-dimensional Brownian motion follows any given continuous path with a given precision up to a given time is positive.

Let $\rho = |z|$. Now if $|B_\tau| = 1$ then the Brownian motion $B_t, 0 \leq t \leq \tau$, will hit the circles of radii $r_k = \rho\, 2^k$, $k = 0, 1, 2, \ldots, n_0(\rho)$ centered at 0 where $n_0(\rho)$ is the largest integer n such that $\rho\, 2^n \leq 1$. Denote the hitting times of those circles by T_k, $k = 0, 1, \ldots, n_0(\rho)$. If for some $k = 0, 1, \ldots, n_0(\rho) - 1$, B_t, $t \geq T_k$, makes a loop around 0 before hitting the circles of radii r_{k-1} or r_{k+1}, then the Brownian motion hits ∂D and $|B_\tau| < 1$. Apply the strong Markov property for T_k, $k = 0, 1, \ldots, n_0(\rho) - 1$, to show that

$$\mathsf{P}^z[|B_\tau| = 1] \leq (1-q)^{n_0(\rho)}.$$

Then note that $n_0(\rho) > (\log(1/\rho))/(\log 2) - 1$ and hence the claim holds for $C = 1/(1-q)$ and $\alpha = (\log 1/(1-q))/(\log 2)$. □

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