Chapter 1
Introduction

In this introductory chapter, we look at iterations of conformal maps, random processes such as random walks and statistical physics and try to establish some connections between them.

1.1 Iteration of conformal maps and Schramm–Loewner evolution

We present in this section a discrete process that will approximate a continuous flow that we'll later call a *Loewner flow*. The idea is that a growth process induces a flow of the points of a reference domain (the upper half-plane $\mathbb{H}$, in this case). We assume that the reader has some familiarity with Complex Analysis,[1] although we will remind definitions, for instance, when they are central to our goals.

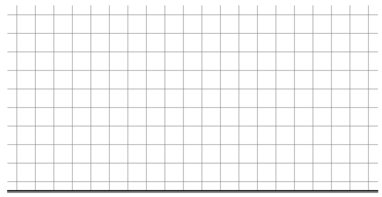
Recall that a differentiable function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is a set containing an open neighborhood of a point z_0, is *conformal* at z_0, if at z_0 the map f preserves the angles in the sense that if P_1 and P_2 are smooth curves that form an angle θ at z_0, then also $f \circ P_1$ and $f \circ P_2$ form an angle θ at $f(z_0)$. If this holds for all points of U, we call f a *conformal map*.

Later we will use a definition that a function is conformal if it is holomorphic and one-to-one. These definitions are equivalent.

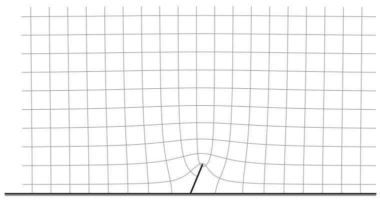
Let $\mathbb{H}$ be the upper half-plane and let $f_k : \mathbb{H} \to \mathbb{H}$ be a sequence of conformal maps where $k \in \mathbb{Z}_{>0}$. Define

$$f^{[1,2,\ldots,n]}(z) = f_1 \circ f_2 \circ \ldots \circ f_n(z). \tag{1.1}$$

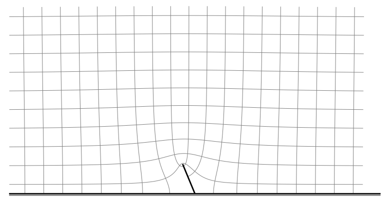
[1] In particular, we assume that the reader is familiar with holomorphic functions, the Cauchy theorem, harmonic functions, the Poisson integral and conformal mappings on the level of Rudin's book [7].



(a) The grid lines in $\mathbb{H}$.



(b) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{R}}(z)$.



(c) The gridlines transformed by the mapping $z \mapsto \hat{f}_{\mathrm{L}}(z)$.

Fig. 1.1 Two elementary conformal transformations that are being iterated in the process illustrated in Figure 1.2. Grid lines can be used to illustrate the action of conformal maps.

Suppose that each f_k maps $\mathbb{H}$ onto a set which is the complement (with respect to $\mathbb{H}$) of a bounded set, whose boundary is a curve, and suppose that $|f_k(z)| \to \infty$ as $|z| \to \infty$. Then it turns out that f_k extends continuously to the boundary of $\mathbb{H}$.

If the bounded set is a line segment, say, then the line segment starts from the boundary, from the point that is the base point of the line segment, and ends to an interior point, which is the tip of the line segment. Thus we can talk about the point $x_k \in \mathbb{R}$ that is mapped to the tip by f_k. Let ξ_k be the base point of the line segment. Define now $\hat{f}_k(z) = f_k(z + x_k) - \xi_k$. Then $\hat{f}_k$ is conformal and it maps $\mathbb{H}$ onto the complement of a line segment, whose base point is 0, it maps ∞ to ∞ and 0 to the tip of the line segment. It turns out (we will come back to this in the later chapters) that it is useful to consider conformal maps that for large values of $|z|$ are close to identity, in the sense that they neither expand or shrink the grid as in Figure 1.1 far away from the origin.

If we iterate maps of this form, for instance, $\hat{f}_1 \circ \hat{f}_2$, then the composition will be a map from $\mathbb{H}$ onto the complement of a piecewise smooth curve. The continuity of the curve where the (images of) line segments meet follows basically from the fact that 0 is the base point of $\hat{f}_2$ and 0 is mapped to the tip point of $\hat{f}_1$ by $\hat{f}_1$.

Figure 1.2 illustrates how the iterates $\hat{f}^{[1,2,\ldots,n]}$ look like. In that case, we have chosen two conformal maps (see Figure 1.1) that correspond to the line segments of the same length forming angles $\alpha\pi$ and $(1-\alpha)\pi$ with the positive real axis. Each f_k is one of the two maps.

The above construction is discrete in nature. The parameter n acts naturally as time of the growth process. If we wish study a continuous time limit of the iterates $\hat{f}^{[1,2,\ldots,n]}$, we need to take large n and adjust the elementary conformal maps so that the sizes of the line segments are small, but the composed piecewise smooth curve reaches roughly to a constant height. This can be achieved by considering $\hat{\phi}_k(z) = n^{-a} \hat{f}_k(n^a z)$ where $a > 0$ is suitably chosen constant. We will in later chapters verify this claim.

Let $F_t^{(n)}$ denote the iterate $\hat{\phi}^{[1,2,\ldots,\lfloor nt \rfloor]}$ for any $n \in \mathbb{Z}_{>0}$ and $t \in [0,1]$.[2] When n is large, the composed piecewise smooth curve corresponding to $\hat{\phi}^{[1,2,\ldots,\lfloor nt \rfloor]}$ increases by tiny steps as t is increased. It seems reasonable to expect that the limit $\lim_{n\to\infty} F_t^{(n)}$ exists and defines a continuous-time flow of the points of $\mathbb{H}$.[3] This is indeed the case at least when the sequence f_k are random, symmetrically distributed[4] and independent.

These continuous-time flows are in general called *Loewner flows* and the particular case of a random, symmetric and independent sequence will be a *Schramm–Loewner evolution*.

1.2 On stochastic models and connection to statistical physics

1.2.1 Random walk and Brownian motion

We also assume some familiarity with Probability Theory.[5] In this section we wish to give some examples of stochastic processes that we encounter in this book.

Recall that a *stochastic process* is a collection of random variables indexed by a ordered set. The order relation of the index set is interpreted as giving the direction of *time*.

As a concrete example let us consider the *simple random walk* on $\mathbb{Z}$. Fix some $x \in \mathbb{Z}$. We will denote probability measures generally by P. Let X_k, $k \in \mathbb{Z}_{>0}$, be a sequence of random variables which take two possible values ± 1, that is,

$$\mathsf{P}[X_k \in \{-1,+1\}] = 1.$$

Assume that X_k, $k \in \mathbb{Z}_{>0}$, are independent.[6]

The formula

$$S_t = x + \sum_{k=1}^{t} X_k \tag{1.2}$$

[2] We use a common notion that $\lfloor x \rfloor$ is the largest integer smaller or equal to x.

[3] Such a limit is an example of *scaling limit*. Two typical features of a scaling limit are that there are scaling factor involved, such as n^{-a} and n^a above, which ensure that the limit exists, and that the limiting object will be described by continuous variables (another term is a continuum limit).

[4] Symmetrically distributed means here that $\hat{f}_{\mathrm{R}}$ and $\hat{f}_{\mathrm{L}}$ are equally likely.

[5] Preferably, measure theoretic probability on the level of Durrett's book [3].

[6] As a reminder, for this particular type of random variables, X_k, $k \in \mathbb{Z}_{>0}$, are independent if it holds that $\mathsf{P}[X_k = x_k \text{ for all } k \in [\![1,n]\!]] = \prod_{k \in [\![1,n]\!]} \mathsf{P}[X_k = x_k]$ for any $n \in \mathbb{Z}_{>0}$ and for any $x_1, x_2, \ldots, x_n \in \{-1,+1\}$.

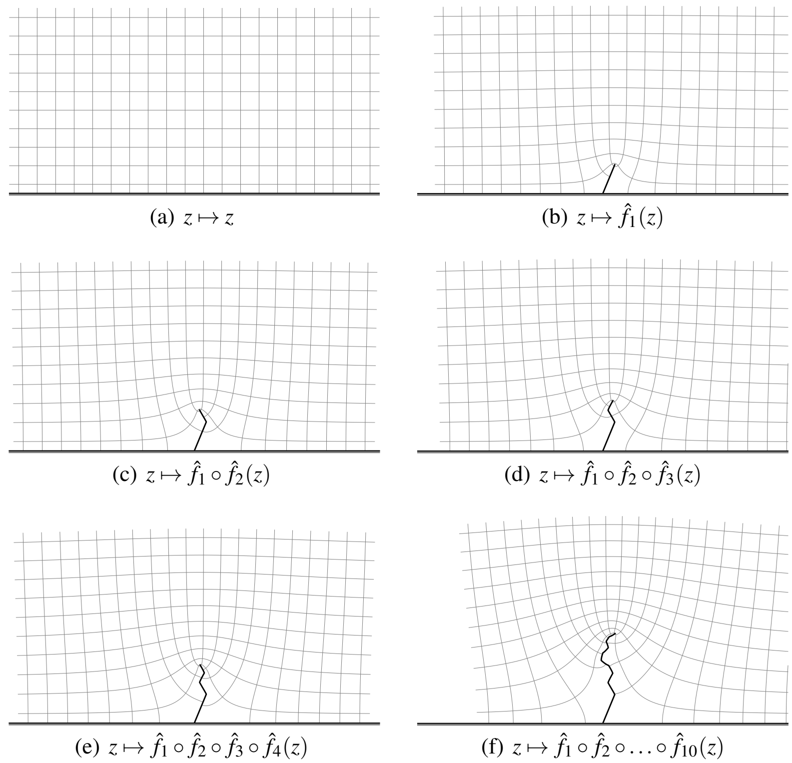


Fig. 1.2 Consider conformal maps from the upper half-plane onto the complements of line segments. We can arrange so that ∞ is mapped to ∞ and that the base point of the line segment is 0 as well as the point which gets mapped to the tip of the segment. The figures here illustrate how iterations of such maps look like.

defines a stochastic process[7] $(S_t)_{t\in\mathbb{Z}_{\geq 0}}$ called random walk in a discrete time variable t. If the random variables X_k, $k \in \mathbb{Z}_{>0}$, have symmetric distribution, that is, $\mathsf{P}[X_k = -1] = \mathsf{P}[X_k = +1] = \frac{1}{2}$, then the process is called *simple random walk* on $\mathbb{Z}$.

As it is often the case, we wish to derive a continuum limit of the simple random walk. It will be a scaling limit in the same sense as in the previous section. For that purpose, we choose a constant $a > 0$ and consider the continuous-time process $(n^{-a} S_{\lfloor nt \rfloor})_{t\in\mathbb{R}_{\geq 0}}$. It turns out that for suitably chosen constant a this process will converge as $n \to \infty$ to a stochastic process $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ called *Brownian motion*. From the *Central Limit Theorem* (CLT) we know that $a = 1/2$ and that all the finite dimensional distributions, that is, the probability laws of vectors of type $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$, are Gaussian.

The discrete-time flow presented in Section 1.1 can be seen as a kind of random walk on the set of all conformal maps of the upper half-plane. Luckily, there is also

[7] We use the notation $(X_t)_{t\in I}$ where usually $I = \mathbb{Z}_{\geq 0}$ or $I = \mathbb{R}_{\geq 0}$, to denote a stochastic process.

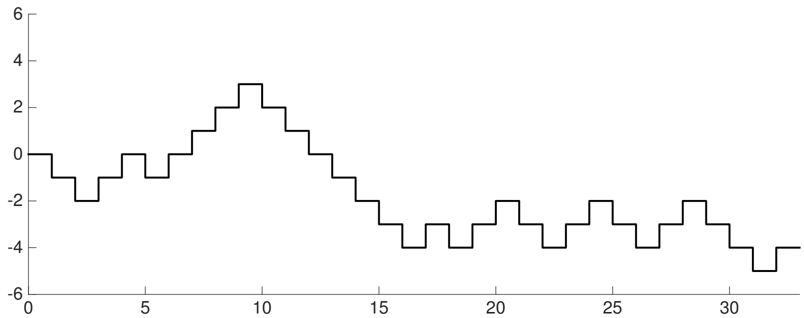


Fig. 1.3 Simple random walk

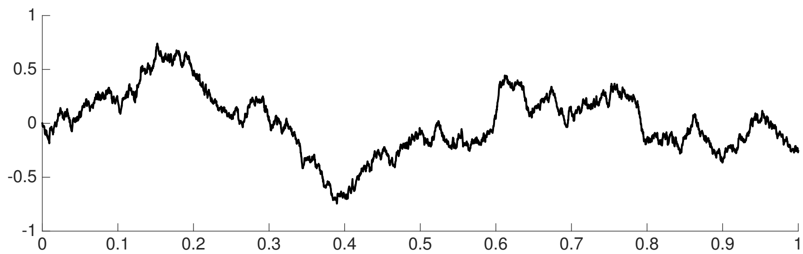


Fig. 1.4 1D Brownian motion

a concrete way to map the Brownian motion to the continuous-time flow, namely, the Loewner equation, and the resulting flow is Schramm–Loewner evolution.

1.2.2 Ising model and other statistical physics models

At this point, it makes sense to ask: why are Schramm–Loewner evolutions needed? What are they good for?

In short, Schramm–Loewner evolutions appear in statistical physics under specific circumstances as *interfaces*, that is, domain walls separating parts of the system which differ in some microscopic property.

A typical example of a lattice model of statistical physics (i.e., a simplified model defined on a lattice such as $\mathbb{Z}^d$) is the *Ising model*, which models ferromagnetic material. Each site v is occupied by an elementary magnet, *spin*, which takes values $\sigma_v \in \{\pm 1\}$. The Ising model is defined by giving a energy functional which in this case is

$$H(\underline{\sigma}) = -\sum \sigma_v \sigma_w.$$

Here $\underline{\sigma} = (\sigma_v)_{v \in V}$ is the spin configuration of the system and V is a finite subset of the square lattice $\mathbb{Z}^2$ (we focus here on two-dimensional model). The sum in H is over neighboring pairs of sites. The more there are pairs of aligned spins, the

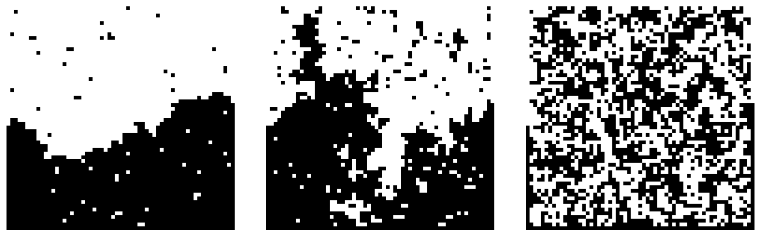


Fig. 1.5 Ising model with Dobrushin boundary conditions for $T < T_c$, $T = T_c$ and $T > T_c$. Here the black pixels are vertices with $\sigma = +1$ and the white pixels are vertices with $\sigma = -1$. An *interface* is a broken line separating white and black regions.

more this functional favours the configuration (that is, the configuration has smaller energy) — this can be seen as the source of the ferromagnetic phenomenon.

In the definition of the Ising model, we take the configuration $\underline{\sigma} \in \{\pm 1\}^V$ to be random. Its law is given by the Boltzmann distribution corresponding to the energy functional H, i.e., the probability of observing $\underline{\sigma}$ is proportional to $\exp(-\beta H(\underline{\sigma}))$. Here $\beta = 1/T$, the inverse temperature, is a parameter.

The behaviour of the systems depends drastically on the temperature T, as the reader can see from Figure 1.5. In the figure we use so called Dobrushin boundary conditions, where we force the spins on the two complementary boundary arcs to be constant -1 on one of them and $+1$ on the other. The *interface* (also known as a domain wall), which is the broken line separating the large $+1$-cluster and the large -1-cluster, can be studied when these boundary conditions are used.

The *scaling limit* of the interface is obtained by fixing a shape, say, a square and the Dobrushin boundary conditions on its boundary and then by approximating that shape by finite subsets of a lattice with a lattice mesh parameter. The scaling limit is the limit as the lattice mesh tends to zero.

The phase transition of the model can be explained in terms of interface in the following way. There is a critical temperature T_c such that for $T < T_c$ for large systems looked far away (i.e. in the scaling limit) the interface is close to the minimal energy line with fluctuations of order $\sqrt{N}$, where N is the side length of the box. As T approaches T_c the fluctuations grow and at T_c they are of the size of the system. Therefore $T = T_c$ is the smallest value of the parameter where we expect a non-trivial scaling limit for the interface. The fact that the scaling limit at $T < T_c$ is non-random is a result of [6]. For $T > T_c$, when looked far away, the spins behave more or less independently of the of each other and the interface looks like the interface of $T = \infty$, for which value the spin configuration is truly totally disordered (independent coin flips).

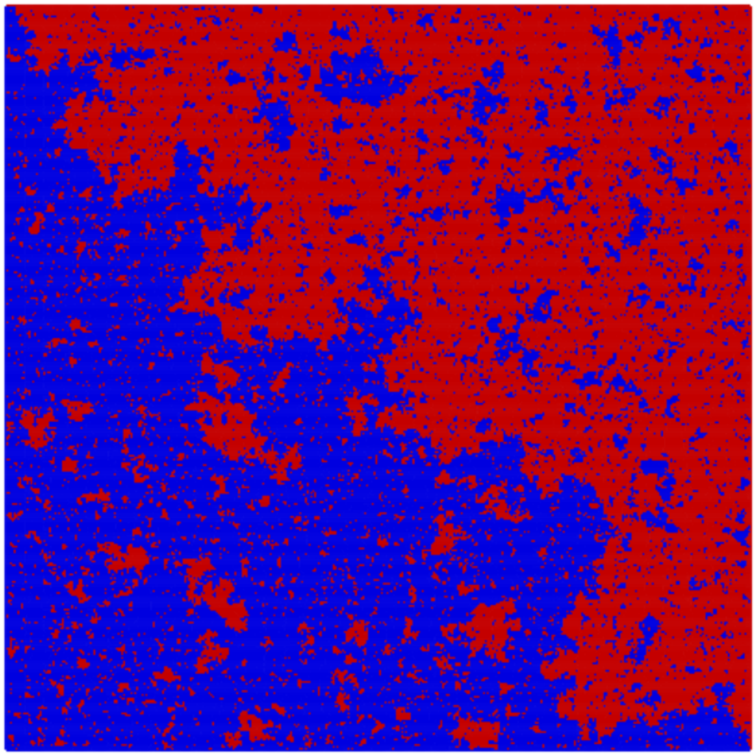


Fig. 1.6 Ising model at criticality

1.2.3 Conformal invariance of the scaling limits

The success of the Schramm–Loewner evolutions is that they give an efficient tool for verifiyng conformal invariance in statistical physics models in the context of random curves and their scaling limits.

Based on physical arguments, the scaling limit is expected to be scale invariant. In fact, under some hypothesis such as partial rotation invariance of the Hamiltonian ($\pi/4$-rotation invariance of the Ising model on $\mathbb{Z}^2$) and short range of the interactions, it is expected that the scaling limit is even conformally invariant. Conformal invariance could be described to be *local* rotation, scale and translation invariance. Here "local" refers to the fact that the factor that we use in scale invariance, for instance, can vary over the domain. Consult, for instance, the introduction of [5] for an introduction to the physical theories of phase transitions.

More concretely, the conformal invariance property of the Ising model should be understood in the following way. If we start from any two shapes (simply connected domains) and approximate both of them with sequences of discrete domains then in the scaling limit the interfaces have laws that are equal, in the sense that they are conformal images of each other.

This property is related to the conformal Markov property of iterates of conformal maps. Namely consider the conditional law of $\hat{f}^{[\![1,n+m]\!]}$ given that we know $\hat{f}_k$, $k \in [\![1,n]\!]$. That conditional law is just the law of $\hat{f}^{[\![m+1,n+m]\!]}$ transformed by the (non-random or known) conformal map $\hat{f}^{[\![1,n]\!]}$. This is an evidence of a connection between statistical physics and the iterates of conformal maps. We call the argument connecting statistical physics and Schramm–Loewner evolutions as *Schramm's principle* and it can be found in [10] or in the original article by Schramm [8].

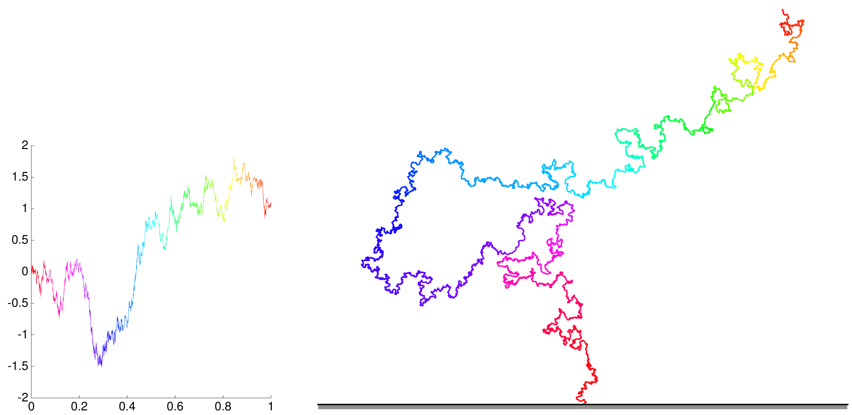


Fig. 1.7 Realizations of a 1D Brownian motion (left) and the corresponding SLE(3) (right) driven by the Brownian motion. SLEs are random fractal curves.

1.3 An example: Cardy's formula

In this section we will give a concrete example with some details that highlight the main topics of this text and the example is one of the main application of the theory of Schramm–Loewner evolution. To be able to present this material we always pick a tool which is the easiest to us even if it isn't fully mathematically justified. We present the full argument later in the text.

1.3.1 Percolation model, crossing events and the critical parameter

Consider the triangular lattice which is formed by the centers of the regular hexagonal tiling of the plane. Take a finite, simply connected[8] subgraph of the triangular lattice. We call the centers of the hexagons *sites*.

In the percolation model, each site carries a random variable which takes value *open* or *closed*. In any pictures, we color the corresponding hexagon green if the site is open and red if the site is closed. From the modelling perspective, the open sites represent channels through which a substance, say, water can flow. Therefore if we inject water into the sites of a set A_1, the water will flow to all the sites connected by a path of open sites to A_1. In particular we are interested in connection events that for fixed A_1 and A_2 there exists a connected path from A_1 to A_2 that stays in a set B. We denote this event by $A_1 \leftrightarrow_B A_2$.

The decision, whether a site is open or closed, is made randomly, independently and from the same distribution at each site. This leaves only one parameter in the model, which is the quantity $p := \mathsf{P}[\text{the site } x \text{ is open}] \in [0,1]$ which doesn't depend on x. Let us denote the law of the percolation of configuration by μ_p. It is a product measure over the sites.

Let us concentrate on a *crossing probability*

$$\mu_p(A_1 \leftrightarrow_B A_2)$$

as a function of p. Here A_1, A_2, B are sets of sites on the triangular lattice and the event $A_1 \leftrightarrow_B A_2$ occurs if and only if there is a connected path of open sites that stays within B and the two endpoints are in A_1 and A_2.

Consider first the crossing probability for a fixed shape with varying size. More specifically, take rhombi $R_N = \{x\mathbf{e}_1 + y\mathbf{e}_2 : x,y \in [\![1,N]\!]\}$ where $\mathbf{e}_1 = 1$ and $\mathbf{e}_2 = \exp(\mathrm{i}\pi/3)$ are two vectors in the plane that generate the triangular lattice. Denote by $f(p,N)$ the probability of $A_1 \leftrightarrow_B A_2$ under μ_p, when $B = R_N$ and A_1 and A_2 are the left and right edges of R_N The probability of the occurrence of a left-to-right crossing in R_N by a path of open sites is a monotonically increasing function of p for fixed N. As illustrated in Figure 1.8, as N tends to infinity the crossing probability tends to a sharp step function. More accurately

$$\lim_{N\to\infty} f(p,N) = \begin{cases} 0 & \text{, when } p < \frac{1}{2} \\ \frac{1}{2} & \text{, when } p = \frac{1}{2} \\ 1 & \text{, when } p > \frac{1}{2} \end{cases}. \tag{1.3}$$

We would arrive to a similar conclusion if we had taken a rhombus with a different aspect ratio. The only difference is that the limit of the crossing probability at $p = 1/2$ is not necessarily $1/2$, but it can take some other value in $(0,1)$. The parameter

[8] Simply connectedness means that the corresponding domain consisting of hexagons is a simply connected domain, i.e., the domain doesn't have any holes — in other words, if we have a closed path of hexagons in the domain, it cannot disconnect any point in the complement of the domain from infinity.

$p = 1/2$ is critical in the sense that outside criticality the limit is trivial, either 0 or 1, but at $p = 1/2$, there is a non-trivial limit.

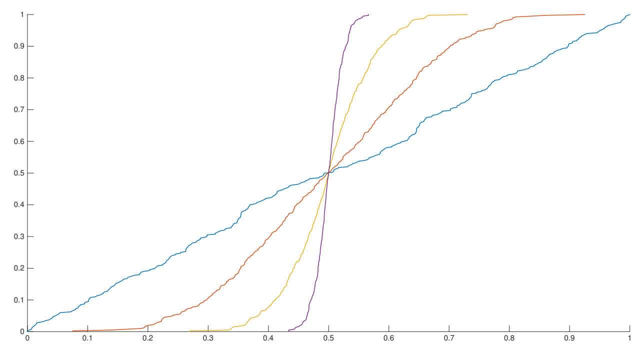


Fig. 1.8 The crossing probabilities of a left-to-right crossing in a rhombus R_N of side length N. The crossing probability is estimated using a computer simulation and plotted as a function of p for different values of N ($N = 1$ blue, $N = 4$ orange, $N = 16$ yellow, $N = 64$ purple). Different values of p are coupled using standard approach using uniform random variables and the sample size is 200 for each value of N.

In fact, the limit is also non-trivial in the sense that it depends non-trivially on the aspect ratio of the rhombus. In the next subsection we will try to see how to derive a formula for it using conformal invariance hypothesis.

1.3.2 Cardy's formula from SLE(6)

Consider for simplicity the probability in a rectangle u, v, i, w for an open crossing from wu to vi. Map the rectangle u, v, i, w conformally onto the upper half-plane $\mathbb{H}$ such that $u \mapsto U_0, v \mapsto V_0, w \mapsto W_0, i \mapsto \infty$. The exact form of the mapping doesn't play a role here.

Next introduce a new layer of hexagons around the rectangle, as in Figure 1.10. Assign boundary conditions such that the hexagons on uv and vi are closed and on iw and wu open. Then there will be an interface separating the closed cluster and the open cluster that touch the boundary. In Figure 1.10, this is the blue path.

We can read the crossing event from the interface. Namely, the left-to-right crossing exists if and only if the interface hits vi before iw.

Introduce also a conditional version of the conformal map. Suppose that the interface is $\gamma(t)$, $t \in [0, T]$. The conditional probability of an open crossing given $\gamma(s)$, $s \in [0, t]$, is again a crossing probability but now in the complement of $\gamma[0, t]$ in the rectangle from the union of wu and the left-hand side of $\gamma[0, t]$ to vi. Therefore it is natural to transform that domain onto the upper half-plane and take $\gamma(t) \mapsto U_t$, $v \mapsto V_t$, $i \mapsto \infty$, $w \mapsto W_t$.

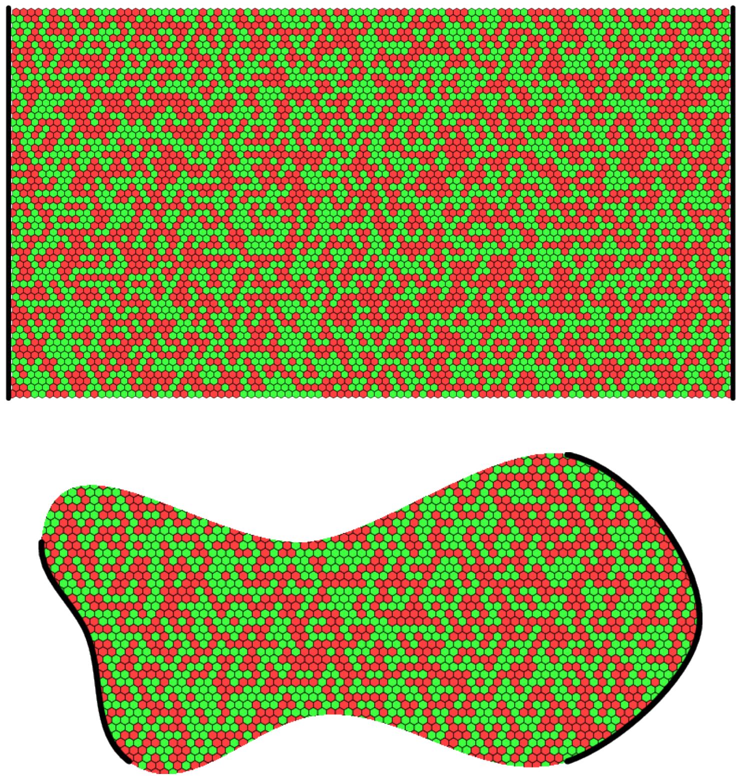


Fig. 1.9 Percolation on two different shapes. Cardy's formula tells that the probability of an open crossing of the quadrilaterals depends only on the conformal modulus in the scaling limit, at criticality, and gives an explicit expression for it.

Now we make an assumption that the scaling limit of the interface is one of the conformally invariant curves and we make a guess[9] the scaling limit is $\mathrm{SLE}(6)$. Under some further assumptions (that the conformal maps are nearly the same around i and that the curve is parametrized suitably) it holds that

$$U_t = \sqrt{6}B_t, \qquad \dot{V}_t = \frac{2}{V_t - U_t}, \qquad \dot{W}_t = \frac{2}{W_t - U_t}.$$

The first equality is the fact that the process is $\mathrm{SLE}(6)$ and the two others are Loewner equations.

[9] There are some reasons to expect that the scaling limit of a percolation interface is $\mathrm{SLE}(6)$. Most importantly, it has a *locality property* that it shares with $\mathrm{SLE}(6)$ which means that the law of the curve is not affected by perturbing the domain away from the curve.

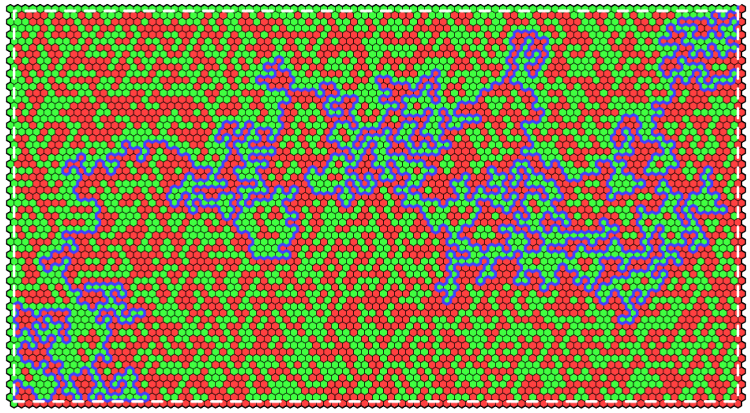


Fig. 1.10 After introducing the extra layer of hexagons for boundary conditions, there will be interface that separates the red and blue clusters that touch the boundary.

Set $Z_t = (U_t - W_t)/(V_t - W_t)$, which is a conformal invariant (in this form invariant under scaling and translation) called cross-ratio. That is equivalent of mapping $\mathbb{H}$ with marked points U_t, V_t, ∞, W_t onto $\mathbb{H}$ with marked points $z, 1, \infty, 0$. We further map the latter domain using a conformal map of the form $\phi(z) = C\int^z w^{-2/3}(1-w)^{-2/3}\mathrm{d}w$ onto a equilateral triangle PQR. Suppose that $\phi(W) = P$, $\phi(V) = Q$ and ϕ. Then $\zeta_t := \phi(Z_t) \in PQ$.

A based on stochastic calculus we can verify that the process ζ_t is a time change of a Brownian motion on PQ and thus the crossing probability, which can be formulated as the probability that the process ζ_t hits Q before P, can be calculated. After an argument from stochastics (time-changed Brownian motions are conserved on average) and some algebra we end up to famous *Cardy's formula*

$$\mathsf{P}[\alpha_1 \overset{\text{open}}{\longleftrightarrow} \alpha_2] = \frac{\phi(z)-\phi(0)}{\phi(1)-\phi(0)} = \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} z^{\frac{1}{3}}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};z\right)$$

where $z = Z_0$, Γ is the gamma function and ${}_2F_1$ is the hypergeometric function. The original articles on Cardy's formula are [2, 4, 9, 11].

1.4 On reading this book

The next two chapters review some background material on Stochastic Calculus and Complex Analysis. If the reader is familiar with those topics, she/he can merely browse through them and move directly to the main chapters, Chapters 4–6. Those chapters build on the prior chapters and should be read preferably in the order of presentation. In addition, appendices with additional material and exercises are provided in separate documents.

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