Appendix C
Supplementary material on complex analysis

C.1 Basics of complex and harmonic function theory

Let us start by a warmup which should be familiar, for a reader with background in elementary complex analysis.

Define differential operators, *complex derivative* $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, *conjugate complex derivative* $\bar{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ and the Laplace operator $\Delta = \partial_{xx} + \partial_{yy} = 4\bar{\partial}\partial$. Remember that sufficiently differentiable functions $f : U \to \mathbb{C}$ and $h : U \to \mathbb{R}$ are *holomorphic* and *harmonic* at z_0, respectively, if

$$\bar{\partial} f = 0 \tag{C.1}$$

and

$$\Delta h = 0 \tag{C.2}$$

in a neighborhood of z_0. Harmonic and holomorphic functions are known to be infinitely differentiable. Any partial derivative of h, such as $\partial_x h, \partial_{yy} h, \partial_{xxy} h, \ldots$, is harmonic as well as any complex derivative of f, namely $f', f'', f''', \ldots$ where $f' = \partial f$, is holomorphic, because the differential operators with constant coefficients commute.

The real and imaginary parts of a holomorphic function are harmonic. In fact we can extend the definition of harmonicity to complex valued functions. Then any holomorphic function is harmonic. Conversely for any (real) harmonic function h, we can find a holomorphic function f such that $\operatorname{Re} f = h$. The function $g = \operatorname{Im} h$ is called a harmonic conjugate of g.

Holomorphic functions are complex analytic in the sense that if f is holomorphic in $B = B(z_0, r)$ then there exists a sequence of coefficients $c_n \in \mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$, such that for all $z \in B$

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots \tag{C.3}$$

The series converges absolutely in B. There are many direct consequences of this fact. One example is the following.

Theorem C.1 (Maximum modulus principle). *Let f be holomorphic on U. If $\overline{B(z_0,r)} \subset U$, then $|f(z_0)| \le \max_{\theta\in[0,2\pi]} |f(z_0+re^{\mathrm{i}\theta})|$. Moreover, equality holds only if f is a constant function.*

Proof. Since the power series $f(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots$ converges absolutely, we can change the order of sums and integration and show that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|f(z_0+re^{\mathrm{i}\theta})\right|^2 \mathrm{d}\theta &= \sum_{k,l=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi} c_k\overline{c_l}r^{k+l}e^{\mathrm{i}(k-l)\theta}\mathrm{d}\theta \\ &= \sum_{k=0}^{\infty}|c_k|^2 r^{2k}\end{aligned}$$

Since $|c_0|^2 = |f(z_0)|^2$, it follows that

$$|f(z_0)|^2 \le \max_{\theta\in[0,2\pi]}\left|f(z_0+re^{\mathrm{i}\theta})\right|^2$$

where the equality holds only if $c_k = 0$ for all $k > 0$. □

The following result is a direct consequence of the above theorem. We leave the proof to the reader.

Theorem C.2 (Schwarz lemma). *Suppose f is holomorphic on $\mathbb{D}$, $\|f\|_\infty \le 1$ and $f(0) = 0$. Then*

$$\begin{aligned}|f(z)| &\le |z| \quad \text{for any } z\in\mathbb{D}, \\ |f'(0)| &\le 1.\end{aligned}$$

If equality holds for some z in the first inequality or equality holds in the second inequality, then $f(z) = \lambda z$ for each $z \in \mathbb{D}$, where λ is a constant and $|\lambda| = 1$.

C.1.1 Möbius maps

Definition C.1. A mapping of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d\in\mathbb{C}, \qquad ad-bc\neq 0$$

is called a *Möbius map* or *fractional linear transformation*.

If $ad-bc = 0$, then either the denominator is identically zero and hence the fraction is not well-defined or the numerator and denominator are constant multiples of each other and hence the fraction is constant. Therefore these maps make sense only when $ad-bc\neq 0$.

It is possible to extend the domain of definition and the range of any Möbius map to the entire *Riemann sphere* (also known as *extended complex plane*) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Thus a way to describe Möbius maps is that they are the conformal self-maps of $\hat{\mathbb{C}}$.

Proposition C.1. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is Möbius map, then f is one-to-one and onto and the inverse is Möbius map.*

This follows from the following, result which is left as an exercise.

Lemma C.1. *If $A \in \mathbb{C}^{2\times 2}$ is invertible ($\det A \neq 0$), define a Möbius map by*

$$\phi_A(z) = \frac{a_{11}\, z + a_{12}}{a_{21}\, z + a_{22}}$$

where $A_{ij} = a_{ij}$. For any invertible $A, B \in \mathbb{C}^{2\times 2}$, it holds that

$$\phi_A \circ \phi_B = \phi_{AB}.$$

Consequently, the inverse map of any Möbius map is given by $\phi_A^{-1} = \phi_{A^{-1}}$.

We also leave as an exercise to show the following.

Lemma C.2. *A map $\phi : \mathbb{D} \to \mathbb{D}$ is conformal and onto if and only if*

$$\phi(z) = e^{i\theta}\, \frac{z - a}{1 - \bar{a}z} \tag{C.4}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

C.2 Continuity up to the boundary

A compact set $A \subset \mathbb{C}$ is said to be *locally connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any two points $a, b \in A$ with $|a - b| < \delta$, there exist a closed connected set B with $a, b \in B \subset A$ and $\operatorname{diam} B < \varepsilon$. For non-bounded closed $A \subset \hat{\mathbb{C}}$, we could adjust this definition and the next theorem by defining metric on the Riemann sphere $\hat{\mathbb{C}}$ that makes $\hat{\mathbb{C}}$ a compact space.

In topology, a compact connected set which contains more than one point, is called a continuum.

Theorem C.3. *Let $U \subset \mathbb{C}$ be a bounded domain. A conformal onto map $f : \mathbb{D} \to U$ extends continuously to $\mathbb{D} \cup \partial\mathbb{D}$ if and only if ∂U is locally connected.*

Proof. We claim that the following are equivalent

1. $f : \mathbb{D} \to \Omega$ extends continuously to $\overline{\mathbb{D}}$
2. ∂U is closed curve
3. ∂U is locally connected

4. $\mathbb{C}\setminus U$ is locally connected .

The implications 1.⇒2. and 2.⇒3. are obvious.

Suppose that 3. holds. Let δ and ε be as in the definition of locally connected sets which holds for ∂U. We can assume that $\delta < \varepsilon$. Let $a,b \in \mathbb{C}\setminus U$ with $a \neq b$ and $|a-b| < \delta$. If $[a,b] \subset \mathbb{C}\setminus U$ then clearly a and b can be connected with a path with diameter less than δ. Suppose that $[a,b]$ intersect ∂U and that a_1 and b_1 are the first and last points on $[a,b] \cap \partial U$. Then $|a_1 - b_1| < \delta$, and by the assumption, a_1 and b_1 can be connected by a continuum in ∂U with diameter less than ε. Consequently a and 1 can be connected in $\mathbb{C}\setminus U$ by a continuum of diameter less than 3ε. Thus 4. holds and implication 3.⇒4. follows.

Assume now that 4. holds. Let δ and ε be as in the definition of locally connected sets which holds for $\mathbb{C}\setminus U$. We can assume that $\delta < \varepsilon$. Suppose that $f(0) = 0$ and $B(0,R_1) \subset \Omega \subset B(0,R_2)$. Let $\delta_1 > 0$. To show that f is continuous upto to the boundary we need to show that f is uniformly continuous near the boundary. Consequently, it is enough to take $z_1, z_2 \in \mathbb{D}$ such that for some $z_0 \in \partial\mathbb{D}$ it holds that $z_k \in B(z_0,\varepsilon)$, $k = 1,2$. By Lemma 4.7, it holds for some $\delta_2 \in [\delta_1, \sqrt{\delta_1}]$ that the length of $\overline{C}$, where $C = f(\{z \in \mathbb{D} : |z - z_0| = \delta_2\})$, is at most $K(R_2)/\sqrt{\log(1/\delta_1)} < \delta$. Let the end points of C be a and b. Then by the assumption, a and b can be connected by a continuum C_1 with diameter less than ε. Thus a and b are separated from infinity in $\mathbb{C}$ by $C \cup C_1$ when $\varepsilon < R_1$. Since $C \cup C_1 \subset B(a,\varepsilon)$, it follows that

$$|f(z_1) - f(z_2)| < 2\varepsilon.$$

The implication 4.⇒1. follows. □

C.3 Schwarz–Christoffel maps

Theorem C.4. *Let U be the interior of a polygon γ with vertices $w_1, w_2, \ldots, w_n$ and interior angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$. Then any conformal and onto map $f : \mathbb{H} \to U$ with $f(\infty) = w_n$ is of the form*

$$f(z) = C_1 + C_2 \int^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} \mathrm{d}\zeta$$

where C_1 and C_2 are constants and $w_k = f(z_k)$, $k = 1,2,\ldots,n-1$.

Proof. We follow here [3]. For simplicity, suppose that w_n is a trivial vertex so that $\alpha_n = 1$.

Observe first that locally near z_k, $k < n$, it holds that $(f(z) - w_k)^{1/\alpha_k} = (z - z_k)\tilde{\phi}(z)$ where $\tilde{\phi}$ is holomorphic function that is non-zero near z_k. Thus $f'(z) = (z - z_k)^{\alpha_k - 1}\phi(z)$ near z_k where $\phi(z)$ is holomorphic and non-zero near z_k.

Next we use Schwarz's reflection principle to extend f to the lower half-plane by continuing across any of the intervals (z_{k-1}, z_k), $k = 1,2,\ldots,n$. Here we set

$z_0 = z_n$. Continue extending the function analytically from lower half-plane to the upper half-plane and back through the intervals. The function obtained is a multi-valued function on a branching cover of $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_{n-1}\}$. Since even number of Schwarz reflections brings the polygon to its original orientation possibly rotated and translated, it holds that on any cover of $\mathbb{H}$, the extension of f is of the form $C_1 f + C_2$ for some constants C_1 and C_2. Consequently, it holds that

$$\frac{(C_1 f(z) + C_2)''}{(C_1 f(z) + C_2)'} = \frac{f''(z)}{f'(z)}.$$

Therefore f''/f' is single valued in $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_{n-1}\}$.

Furthermore, the function $f''/f' = (\log f')'$ has first order poles at z_k with residue $\alpha_k - 1$ for $k = 1, 2, \ldots, n-1$. Thus $z \mapsto f''(z)/f'(z) - \sum_{k=1}^{n-1} (\alpha_k - 1)/(z - z_k)$ is an entire function. By the assumption f is holomorphic around ∞. Thus $f(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$ as $z \to \infty$. It follows that $f''(z)/f'(z)$ tends to zero as $z \to \infty$. Therefore $f''(z)/f'(z) - \sum_{k=1}^{n-1} (\alpha_k - 1)/(z - z_k) = 0$ identically. The claim follows by integrating twice. $\square$

References

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