Appendix B
Supplementary material on stochastic calculus

B.1 Usual conditions

Let's comment on some assumptions usually assumed in textbooks on stochastic analysis. If we are given a probability space $(\Omega, \mathscr{F}', \mathsf{P})$ and a filtration $(\mathscr{F}_t')_{t\in\mathbb{R}}$, then we can complete $\mathscr{F}'$ by including all null sets and use the *usual augmentation* of $(\mathscr{F}_t')_{t\in\mathbb{R}}$ which is defined by including all the null sets in the filtration and making the filtration right continuous:

$$\begin{aligned}
\mathscr{N} &= \{A \subset \Omega : A \subset E \text{ for some } E \in \mathscr{F} \text{ s.t. } \mathsf{P}[E] = 0\} \\
\mathscr{F} &= \sigma(\mathscr{F}' \cup \mathscr{N}) \\
\overline{\mathscr{F}}_t &= \sigma(\mathscr{F}_t' \cup \mathscr{N}) \\
\mathscr{F}_t &= \bigcap_{s>t} \overline{\mathscr{F}}_s.
\end{aligned}$$

The filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ constructed in this way is *right-continuous* in the sense that $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$.

We will assume that $\mathscr{F}$ is complete and that $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ satisfies the *usual conditions*, i.e., it is complete and right-continuous. The right-continuity of the filtration affects the set of stopping times. Here is an example result.

Lemma B.1. *If $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous and $(X_t)_{t\in\mathbb{R}_+}$ is a continuous, adapted $\mathbb{R}^d$-valued process, then the* hitting-time *of a open or closed set $H \subset \mathbb{R}^d$*

$$\tau_H = \inf\{t \in \mathbb{R}_+ : X_t \in H\}$$

is a stopping time.

B.2 Strong Markov property

For the sake of completeness, let's state the following property of Brownian motion which extends the Markov property of Brownian motion (the property that for each $s \in \mathbb{R}_+$, the process $Y_t = B_{t+s} - B_s$ is a standard Brownian motion independent from $\mathscr{F}_s$).

Theorem B.1 (Strong Markov property). *For any stopping time τ which is almost surely finite, the process $(Y_t)_{t\in\mathbb{R}_+}$ defined by*

$$Y_t = B_{\tau+t} - B_\tau$$

is a standard Brownian motion independent of $\mathscr{F}_\tau$.

Remark B.1. Note that in the independence property, an "infinitesimal peek to the future" is allowed because the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous and hence $\mathscr{F}_\tau = \bigcap_{h>0} \mathscr{F}_{\tau+h}$.

B.3 Lemma on approximation by simple processes in $\mathscr{L}^2$

Proposition B.1. *For each $f \in \mathscr{L}^2$, there exist a sequence of bounded, simple $f_n \in \mathscr{L}^2$ such that*

$$\mathsf{E}\left[\int_0^T (f(t,\cdot) - f_n(t,\cdot))^2 \mathrm{d}t\right] \to 0,$$

i.e. f_n converges to f in $L^2(\mathrm{d}t \times \mathrm{dP})$.

Remark B.2. The first and last of the three steps of the proof below are the most important, because we will mostly only consider continuous processes as integrands.

Proof (a sketch). Bounded continuous $f \in \mathscr{L}^2$: Take any sequence of partitions π_n such that $\mathrm{mesh}(\pi_n) \to 0$ as $n \to \infty$ and define a sequence of bounded, simple processes $f_n \in \mathscr{L}^2$ as

$$f_n(t,\omega) = \sum_{k=0}^{m(\pi)-1} f(t_k,\omega)\mathbb{1}_{[t_k,t_{k+1})}(t)$$

when π_n is $0 = t_0 < t_1 < \ldots < t_{m(\pi_n)} = T$. Then

$$\sup_{t\in[0,T]} |f(t,\omega) - f_n(t,\omega)| \leq \sup_{s,t\in[0,T]\,:\,|s-t|\leq \mathrm{mesh}(\pi_n)} |f(t,\omega) - f(s,\omega)|$$

By continuity the right-hand side tends to zero almost surely. Since $|f| \leq C < \infty$ for some constant C, we can apply the dominated convergence theorem (DCT) to show that the right-hand side goes to zero also in $L^2(\mathrm{dP})$. Hence

$$\mathsf{E}\left[\int_0^T |f(t,\cdot)-f_n(t,\cdot)|^2 \mathrm{d}t\right] \le \mathsf{E}\left[T \sup_{t\in[0,T]} |f(t,\cdot)-f_n(t,\cdot)|^2\right] \to 0.$$

Bounded $g \in \mathscr{L}^2$: Take a sequence of continuous functions $\psi_n : \mathbb{R} \to \mathbb{R}$ such that (i) $\psi_n \ge 0$, (ii) $\psi_n(x) = 0$ when $x \notin (-1/n, 0)$ and (iii) $\int_{-\infty}^{\infty} \psi_n(x) = 1$. Define a sequence of bounded, continuous processes $g_n \in \mathscr{L}^2$ as

$$g_n(t,\omega) = \int_0^t \psi_n(s-t) g(s,\omega) \mathrm{d}s$$

The sequence (ψ_n) forms an *approximate identity* and by standard properties of such sequences,

$$\int_0^T (g_n(t,\omega) - g(t,\omega))^2 \mathrm{d}t \to 0.$$

We omit the details of the measurability requirements of $\mathscr{L}^2$. By DCT, $g_n \to g$ in $L^2(\mathrm{d}t \times \mathrm{d}\mathsf{P})$.

General $h \in \mathscr{L}^2$: Define a sequence of bounded processes $h_n \in \mathscr{L}^2$ as

$$h_n(t,\omega) = \begin{cases} -n & \text{if } h(t,\omega) < -n \\ h(t,\omega) & \text{if } h(t,\omega) \in [-n, n] \\ n & \text{if } h(t,\omega) > n \end{cases}$$

Then by DCT, $h_n \to h$ in $L^2(\mathrm{d}t \times \mathrm{d}\mathsf{P})$. □

B.4 Lemma on pathwise interpretation of stochastic integral

We prove next an “obvious” result which allows a kind of a pathwise interpretation of the Itô integral: if two integrands have the same paths up to a stopping time, then the integrals also agree up to that stopping time.

Proposition B.2. *If τ is a stopping time and $f \in \mathscr{L}^2$ and $g \in \mathscr{L}^2$ processes such that $f(t,\omega) = g(t,\omega)$ for any (t,ω) such that $t \le \tau(\omega)$, then for almost all ω*

$$\int_0^t f \,\mathrm{d}B_s(\omega) = \int_0^t g \,\mathrm{d}B_s(\omega)$$

for all $t \le \tau(\omega)$.

Proof. Let $X_t = \int_0^t f \,\mathrm{d}B_s$. It is clearly enough to prove that if τ is a stopping time and $f(t,\omega) = 0$ for $t \le \tau(\omega)$, then for almost all ω, $X_t(\omega) = 0$ for all $t \le \tau(\omega)$.

Assume for a moment that $|f| \le K$. Pick a sequence of simple $f_n \in \mathscr{L}^2$ converging to f in $L^2(\mathrm{d}t \times \mathrm{d}\mathsf{P})$. We can assume that $|f_n| \le K$. Write

$$f_n(t,\omega) = \sum_{k=0}^{m_n-1} a_k^{(n)}(\omega) \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t)$$

Since it is possible that $f_n(t,\omega) \neq 0$ for some (t,ω) satisfying $t \leq \tau(\omega)$, we modify f_n by setting

$$\tilde{f}_n(t,\omega) = \sum_{k=0}^{m_n-1} a_k^{(n)}(\omega) \mathbb{1}_{\left\{\tau < t_k^{(n)}\right\}}(\omega) \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t).$$

Notice that $\tilde{f}_n \in \mathscr{L}^2$ (here we need that τ is a stopping time). Now since $f_n \mathbb{1}_{[\tau,\infty)} \to f \mathbb{1}_{[\tau,\infty)} = f$ in $L^2(\mathrm{d}t \times \mathrm{dP})$, to check that $\tilde{f}_n \to f$ in $L^2(\mathrm{d}t \times \mathrm{dP})$ we have to show that $\tilde{f}_n - f_n \mathbb{1}_{[\tau,\infty)} \to 0$ in $L^2(\mathrm{d}t \times \mathrm{dP})$.

Now

$$\begin{aligned}
\left|\tilde{f}_n(t,\cdot) - f_n(t,\cdot)\mathbb{1}_{[\tau,\infty)}(t)\right| &\leq K \sum_{k=0}^{m-1} \left| \mathbb{1}_{\left\{\tau < t_k^{(n)}\right\}} - \mathbb{1}_{\{\tau \leq t\}} \right| \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t) \\
&\leq K \sum_{k=0}^{m-1} \mathbb{1}_{\left\{t_k^{(n)} \leq \tau < t_{k+1}^{(n)}\right\}} \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t)
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathsf{E}\left[\int_0^T \left(\tilde{f}_n - f_n \mathbb{1}_{[\tau,\infty)}\right)^2 \mathrm{d}t\right] &\leq K^2 \sum_{k=0}^{m-1} \mathsf{P}\left[t_k^{(n)} \leq \tau < t_{k+1}^{(n)}\right] \int_0^T \mathbb{1}_{\left[t_k^{(n)}, t_{k+1}^{(n)}\right)}(t)\mathrm{d}t \\
&\leq K^2 \operatorname{mesh}(\pi_n)
\end{aligned}$$

where $\pi_n = \{t_0^{(n)}, \ldots, t_{m_n}^{(n)}\}$. We can assume that $\operatorname{mesh}(\pi_n) \to 0$ by adding points to the partition if necessary. Since $\tilde{f}_n \to f$ in $L^2(\mathrm{d}t \times \mathrm{dP})$ and $\int_0^t \tilde{f}_n(s,\cdot)\mathrm{d}B_s = 0$ for $t \leq \tau$, and since by the proof of Theorem 2.4 there is a subsequence of $\int_0^t \tilde{f}_n(s,\cdot)\mathrm{d}B_s$ that converges almost surely uniformly on $[0,T]$ for $T > 0$, $X_t(\omega) = 0$ almost surely for any $t \leq \tau$. □

B.5 Itô's formula for Brownian motion

Theorem B.2 (Itô's formula for a Brownian motion). *Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\dot{F}, F', F''$ exist and are continuous, where*

$$\dot{F}(t,x) = \frac{\partial F}{\partial t}(t,x), \quad F'(t,x) = \frac{\partial F}{\partial x}(t,x) \quad \text{and } F''(t,x) = \frac{\partial^2 F}{\partial x^2}(t,x).$$

Then almost surely

$$F(t,B_t) = F(0,B_0) + \int_0^t \dot{F}(s,B_s)\mathrm{d}s + \int_0^t F'(s,B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t F''(s,B_s)\mathrm{d}s \quad \text{(B.1)}$$

for any $t \in \mathbb{R}_+$. For the previous equation we will use the shorthand notation

$$\mathrm{d}F(t,B_t) = \dot{F}(t,B_t)\mathrm{d}t + F'(t,B_t)\mathrm{d}B_t + \frac{1}{2}F''(t,B_t)\mathrm{d}t.$$

Proof. We'll prove this claim in the case when $\dot{F}, F', F''$ are compactly supported. The general case follows from this when we set $F_n = F\,h_n$ where $0 \le h_n \le 1$ is a sequence of smooth functions such that $h_n = 1$ in $[0,n] \times [-n,n]$ and 0 in the complement of $[0,n+1] \times [-n-1,n+1]$.

Take a partition π of $[0,t]$ and write a telescoping sum

$$F(t,B_t) - F(0,B_0) = \sum_{k=0}^{m(\pi)-1} (F(t_{k+1},B_{t_{k+1}}) - F(t_k,B_{t_k})).$$

By the mean value theorem

$$\begin{aligned} &F(t_{k+1},B_{t_{k+1}}) - F(t_k,B_{t_k}) = [F(t_{k+1},B_{t_{k+1}}) - F(t_k,B_{t_{k+1}})] + [F(t_k,B_{t_{k+1}}) - F(t_k,B_{t_k})] \\ &= \underbrace{[F(t_{k+1},B_{t_{k+1}}) - F(t_k,B_{t_{k+1}})]}_{=a_k} + \underbrace{F'(t_k,B_{t_k})(B_{t_{k+1}} - B_{t_k})}_{=b_k} + \frac{1}{2}\underbrace{F''(t_k,\eta_k)(B_{t_{k+1}} - B_{t_k})^2}_{=c_k} \end{aligned}$$

where η_k is a $\mathscr{F}_{t_{k+1}}$-measurable random variable that lies on the interval between B_{t_k} and $B_{t_{k+1}}$. Take a sequence of partitions π_n such that $\mathrm{mesh}(\pi_n) \to 0$ as $n \to \infty$. The claim is that the sums $\sum a_k$, $\sum b_k$ and $\sum c_k$ will converge to each of the three integrals in (B.1), respectively. The convergence will be almost sure along suitable subsequences of π_n.

Define for any $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ the following quantities measuring sizes of oscillations

$$\begin{aligned} O_{(B)}(\delta) &= \sup\{|B_s - B_{s'}| : 0 \le s,s' \le t \text{ s.t. } |s-s'| \le \delta\} \\ O_g(\delta,\delta') &= \sup\{|g(s,x) - g(s',x')| : 0 \le s,s' \le t \text{ s.t. } |s-s'| \le \delta \text{ and } x,x' \in \mathbb{R} \text{ s.t. } |x-x'| \le \delta'\} \\ O_{g,B}(\delta) &= O_g\left(\delta, O_{(B)}(\delta)\right). \end{aligned}$$

Note first that by the mean value theorem

$$F(t_{k+1},B_{t_{k+1}}) - F(t_k,B_{t_{k+1}}) = \dot{F}(\rho_k,B_{t_{k+1}})(t_{k+1} - t_k)$$

where $\rho_k \in (t_k,t_{k+1})$ is a random variable. Now

$$\left|\dot{F}(\rho_k,B_{t_{k+1}}) - \dot{F}(t_k,B_{t_k})\right| \le O_{\dot{F},B}(\mathrm{mesh}(\pi_n))$$

and therefore

$$\left|\sum_k \dot{F}(\rho_k,B_{t_{k+1}})(t_{k+1} - t_k) - \sum_k \dot{F}(t_k,B_{t_k})(t_{k+1} - t_k)\right| \le t\,O_{\dot{F},B}(\mathrm{mesh}(\pi_n))$$

which goes to zero almost surely as $\mathrm{mesh}(\pi_n) \to 0$. By almost sure continuity of $t \mapsto \dot{F}(t, B_t)$,

$$\sum_k \dot{F}(t_k, B_{t_k})(t_{k+1} - t_k) \to \int_0^t \dot{F}(s, B_s)\mathrm{d}s$$

almost surely as $\mathrm{mesh}(\pi_n) \to 0$ and hence

$$\sum_k \dot{F}(\rho_k, B_{t_{k+1}})(t_{k+1} - t_k) \to \int_0^t \dot{F}(s, B_s)\mathrm{d}s$$

almost surely as $\mathrm{mesh}(\pi_n) \to 0$ and we have shown the almost sure convergence of $\sum a_k$.

We know from the definition of Itô integral that

$$\sum F'(t_k, B_{t_k})(B_{t_{k+1}} - B_{t_k}) \to \int_0^t F'(s, B_s)\mathrm{d}B_s \tag{B.2}$$

in L^2. Choose a subsequence of π_n (denoted for simplicity still by π_n) such that this convergence is almost sure. This gives the claim for $\sum b_k$.

Finally,

$$\left|\sum (F''(t_k, \eta_k) - F''(t_k, B_{t_k})) \cdot (B_{t_{k+1}} - B_{t_k})^2\right| \leq O_{F'', B}(\mathrm{mesh}(\pi_n)) \sum (B_{t_{k+1}} - B_{t_k})^2$$

Take a subsequence such that $\sum (B_{t_{k+1}} - B_{t_k})^2$ goes to t almost surely as $\mathrm{mesh}(\pi_n) \to 0$. Then the right-hand side goes to zero almost surely. Now the same calculation as for the quadratic variation of Brownian motion shows that

$$\begin{aligned}
&\mathsf{E}\left(\left(\sum F''(t_k, B_{t_k}) \cdot ((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k))\right)^2\right) \\
&= \sum \mathsf{E}\left(F''(t_k, B_{t_k})^2 \cdot ((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k))^2\right) \\
&\leq \|F''\|_\infty^2 \sum \mathsf{E}\left(((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k))^2\right) \\
&= 2\|F''\|_\infty^2 \sum (t_{k+1} - t_k)^2
\end{aligned}$$

which goes to zero. Now take yet another subsequence such that

$$\sum F''(t_k, B_{t_k}) \cdot ((B_{t_{k+1}} - B_{t_k})^2 - (t_{k+1} - t_k)) \to 0$$

almost surely. Then on the event that $t \mapsto F''(t, B_t)$ is continuous,

$$\sum F''(t_k, B_{t_k})((B_{t_{k+1}} - B_{t_k})^2 \to \int_0^t F''(s, B_s)\mathrm{d}s$$

almost surely. Hence along the chosen subsequence

$$\sum F''(t_k, \eta_k)(B_{t_{k+1}} - B_{t_k})^2 \to \int_0^t F''(s, B_s)\mathrm{d}s \tag{B.3}$$

almost surely giving the claim for $\sum c_k$.

Now we have shown that for fixed t, Itô's formula (B.1) holds almost surely. Therefore it holds almost surely for all rational t. Finally, by continuity of both sides in t, it holds almost surely for all t. □

B.6 Quadratic variation of a stochastic integral

Theorem B.3. *For any $f \in \mathscr{L}^2$, the Itô integral $X_t = \int_0^t f \mathrm{d}B_s$ has finite quadratic variation and*

$$V_X^{(2)}(t) = \langle X \rangle_t$$

almost surely for any t.

Proof. Assume first that $f \in \mathscr{L}^2$ is such that the Itô integral $X_t = \int_0^t f \mathrm{d}B_s$ and the quadratic variation $\langle X \rangle_t$ are bounded processes, that is, there exists a constant K such that for almost all ω and for all t, $|X_t(\omega)| \leq K$ and $\langle X \rangle_t \leq K$.

Let $t > 0$ and $\pi = \{0 = t_0 < t_1 < \ldots < t_m = t\}$. Define

$$\Delta_k = (X_{t_{k+1}} - X_{t_k})^2 - \langle X \rangle_{t_{k+1}} + \langle X \rangle_{t_k}$$

and note that

$$\sum_{k=0}^{m-1} \Delta_k = \sum_{k=0}^{m-1} (X_{t_{k+1}} - X_{t_k})^2 - \langle X \rangle_t .$$

Notice also that for any $0 \leq u \leq t_k$, $\mathsf{E}[\Delta_k | \mathscr{F}_u] = 0$ by the martingale increment orthogonality. Therefore

$$\mathsf{E}\left[\left(\sum_{k=0}^{m-1} \Delta_k\right)^2\right] = \sum_{k=0}^{m-1} \mathsf{E}\left[\Delta_k^2\right].$$

By the inequality $(a+b)^2 \leq 2(a^2 + b^2)$,

$$\begin{aligned}\mathsf{E}\left[\left(\sum_{k=0}^{m-1} \Delta_k\right)^2\right] \leq & 2\sum_{k=0}^{m-1} \mathsf{E}\left((X_{t_{k+1}} - X_{t_k})^4\right) \\ & + 2\mathsf{E}[\langle X \rangle_t \sup\{|\langle X \rangle_s - \langle X \rangle_{s'}| : 0 \leq s, s' \leq t, |s - s'| \leq \mathrm{mesh}(\pi)\}].\end{aligned}$$

The second term goes to zero as $\mathrm{mesh}(\pi) \to 0$ by boundedness and continuity of $\langle X \rangle_t$. So it remains to show that

$$\sum_{k=0}^{m-1} \mathsf{E}\left[(X_{t_{k+1}} - X_{t_k})^4\right] \to 0$$

as $\mathrm{mesh}(\pi) \to 0$.

We will first show that

$$\mathsf{E}\left[\left(\sum_{k=0}^{m-1}(X_{t_{k+1}}-X_{t_k})^2\right)^2\right]\leq 6K^4 \tag{B.4}$$

By using the martingale property of $(X_t)_{t\in\mathbb{R}_+}$

$$\sum_{k=j}^{m-1}\mathsf{E}\left[(X_{t_{k+1}}-X_{t_k})^2\middle|\mathscr{F}_{t_j}\right]=\sum_{k=0}^{m-1}\mathsf{E}\left[X_{t_{k+1}}^2-X_{t_k}^2\middle|\mathscr{F}_{t_j}\right]\leq\mathsf{E}\left[X_{t_m}^2\middle|\mathscr{F}_{t_j}\right]\leq K^2$$

and therefore

$$\begin{aligned}
&\sum_{j=0}^{m-1}\sum_{k=j+1}^{m-1}\mathsf{E}\left[(X_{t_{j+1}}-X_{j_k})^2(X_{t_{k+1}}-X_{t_k})^2\right]\\
&=\sum_{j=0}^{m-1}\mathsf{E}\left[(X_{t_{j+1}}-X_{j_k})^2\sum_{k=j+1}^{m-1}\mathsf{E}\left[(X_{t_{k+1}}-X_{t_k})^2\middle|\mathscr{F}_{t_{j+1}}\right]\right]\\
&\leq K^2\sum_{j=0}^{m-1}\mathsf{E}\left[(X_{t_{j+1}}-X_{j_k})^2\right]\leq K^4
\end{aligned}$$

We also have

$$\sum_{k=0}^{m-1}\mathsf{E}\left[(X_{t_{k+1}}-X_{t_k})^4\right]\leq 4K^2\sum_{k=0}^{m-1}\mathsf{E}\left[(X_{t_{k+1}}-X_{t_k})^2\right]\leq 4K^4.$$

The inequality (B.4) follows directly from the last two inequalities.

Now by the Cauchy–Schwarz inequality

$$\begin{aligned}
&\mathsf{E}\left[\sum_{k=0}^{m-1}(X_{t_{k+1}}-X_{t_k})^4\right]\\
\leq&\mathsf{E}\left[\sup\left\{|X_s-X_{s'}|^2\,:\,0\leq s,s'\leq t,\,|s-s'|\leq\mathrm{mesh}(\pi))^2\right\}\sum_{k=0}^{m-1}(X_{t_{k+1}}-X_{t_k})^2\right]\\
\leq&\left[\mathsf{E}\left[\sup\left\{|X_s-X_{s'}|^2\,:\,0\leq s,s'\leq t,\,|s-s'|\leq\mathrm{mesh}(\pi))^2\right\}^2\right]\mathsf{E}\left[\left(\sum_{k=0}^{m-1}(X_{t_{k+1}}-X_{t_k})^2\right)^2\right]\right]^{\frac{1}{2}}
\end{aligned}$$

And the right-hand side goes to zero by continuity of X_t and by the estimate (B.4).

We have now show that when X_t and $\langle X\rangle_t$ are bounded processes then the quadratic variation exists and $V_X^{(2)}(s)=\langle X\rangle_s$. To complete the proof for a general $f\in\mathscr{L}^2$, let

$$\tau_n=\inf\{t\geq 0\,:\,|X_t|\geq n\text{ or }\langle X\rangle_t\geq n\}$$

and use the above argument for $f_n=f\mathbb{1}_{[0,\tau_n]}$ and $X_t^{(n)}=\int_0^t f_n(s,\cdot)\mathrm{d}B_s$. Notice that $X_t^{(n)}=X_t^{\tau_n}$ and that $\tau_n\nearrow\infty$ almost surely. The claim follows from them. □

B.7 When is a semimartingale a martingale?

Lemma B.2. *Let* $\mathrm{d}X_t = \sum g_t^{(k)} \mathrm{d}B_t^{(k)} + f_t \mathrm{d}t$ *be a semimartingale. Then it is a local martingale if and only if almost surely* $f_t = 0$ *for almost all* t.

Proof. Suppose that $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ is a local martingale. Let $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ be a semimartingale such that $M_0 = 0$ and $\mathrm{d}M_t = \mathrm{d}X_t - \sum g_t^{(k)} \mathrm{d}B_t^{(k)} = f_t \mathrm{d}t$. Then $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ is a local martingale and $M_t = \int_0^t f_s \mathrm{d}s$.

We can localize $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ and assume without a loss of generality that for some $K > 0$, it holds that $\int_0^\infty |f_t| \mathrm{d}t \leq K$ almost surely. Then $(M_t)_{t\in\mathbb{R}_{\geq 0}}$ is a bounded martingale. Write for $0 \leq s < t$ using the properties of conditional expected value and martingale property

$$\begin{aligned}\mathsf{E}[(M_t - M_s)^2 | \mathscr{F}_t] &= \mathsf{E}[M_t^2 | \mathscr{F}_s] - 2M_s \mathsf{E}[M_t | \mathscr{F}_s] + M_s^2 \\ &= \mathsf{E}[M_t^2 | \mathscr{F}_s] - M_s^2 = \mathsf{E}[M_t^2 - M_s^2 | \mathscr{F}_s].\end{aligned}$$

This relation is an instance of the “orthogonality of martingale increments.”

Therefore for any $0 = t_0 < t_1 < \ldots < t_n = t$ it holds that

$$\mathsf{E}[M_t^2] = \sum_{k=0}^{n-1} \mathsf{E}[(M_{t_{k+1}} - M_{t_k})^2] = \leq K \mathsf{E}[\max_k |M_{t_{k+1}} - M_{t_k}|]. \tag{B.5}$$

Since $f_t(\omega) \in L^1$ for each ω, $\max_k |M_{t_{k+1}} - M_{t_k}|$ tends to zero pointwise in ω as $\max_k |t_{k+1} - t_k|$ tends to zero. Since $|M_t| \leq K$, the right hand-side of (B.5) tends to zero by the dominated convergence theorem as $\max_k |t_{k+1} - t_k|$ tends to zero. Therefore it follows that $\mathsf{E}[M_t^2] = 0$ for all t.

By the above argument almost surely $M_t = 0$ for all rational t, and by continuity finally, almost surely $M_t = 0$ for all t, whence the claim follows. □

B.8 Stochastic differential equations

Let X_t be an $\mathbb{R}^n$ valued continuous stochastic process and let B_t be a standard m-dimensional Brownian motion. We say that X_t satisfies the *stochastic differential equation* (SDE)

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}B_t$$

with initial condition $X_0 = Z$ if for each t

$$X_t = Z + \int_0^t F(s, X_s)\,\mathrm{d}s + \int_0^t G(s, X_s)\,\mathrm{d}B_s$$

Here $G(s, X_s)\,\mathrm{d}B_s$ is understood as a matrix product so that the i'th component, $1 \leq i \leq n$, is $\sum_{j=1}^m G^{(i,j)}(s, X_s)\,\mathrm{d}B_s^{(j)}$.

Theorem B.4. *Let B_t be m-dimensional Brownian motion and let*

$$\begin{aligned} F &: [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \\ G &: [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m} \end{aligned}$$

be measurable maps. Let Z be $\mathbb{R}^n$-valued square integrable random variable which is independent from $\sigma(B_t, t \in \mathbb{R}+)$. Suppose that

$$\begin{aligned} |F(t,x)| + |G(t,x)| &\leq C(1+|x|) \\ |F(t,x) - F(t,y)| + |G(t,x) - G(t,y)| &\leq D\,|x-y| \end{aligned}$$

where for any matrix A, $|A| = \sqrt{\sum_{i,j} A_{i,j}}$.

Then there exist a unique continuous solution $(X_t)_{t\in[0,T]}$ to the stochastic differential equation

$$X_0 = Z, \qquad \mathrm{d}X_t = F(t,X_t)\mathrm{d}t + G(t,X_t)\mathrm{d}B_t, \qquad t \in [0,T],$$

with the property that X_t is adapted to the filtration $\mathscr{F}_t^{(B,Z)}$ generated by Z and B_s, $s \in [0,t]$. Furthermore

$$\mathsf{E}\left[\int_0^T |X_t|^2 \mathrm{d}t\right] < \infty.$$

Remark B.3. In the time-homogeneous case, $F(t,x) = F(x)$ and $G(t,x) = G(x)$, these solutions X_t are called *diffusions*. Another viewpoint to diffusions is that it is a family of processes with one element for each starting point $x \in \mathbb{R}^n$. The uniqueness of the solution together with the strong Markov property of Brownian motion imply that diffusions have the following *strong Markov property*: $X_{\tau+t}$ conditioned on $\mathscr{F}_\tau$ is distributed as an independent copy of the diffusion $\tilde{X}_t$ send from X_τ.

References

1. Durrett, R.: Stochastic calculus. a practical introduction. CRC (1996)
2. Durrett, R.: Probability: theory and examples, fourth edn. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge (2010)
3. Kallenberg, O.: Foundations of modern probability. Springer Verlag (2002)
4. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus, *Graduate Texts in Mathematics*, vol. 113, second edn. Springer-Verlag, New York, New York, NY (1991)
5. Øksendal, B.: Stochastic differential equations, sixth edn. Universitext. Springer-Verlag, Berlin, Berlin, Heidelberg (2003)
6. Revuz, D., Yor, M.: Continuous martingales and Brownian motion, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 293, third edn. Springer-Verlag, Berlin (1999)
7. Steele, J.M.: Stochastic calculus and financial applications, *Applications of Mathematics (New York)*, vol. 45. Springer-Verlag, New York, New York, NY (2001)