Department of Mathematics and Statistics
Riemannian geometry
Exercise 5
23.2.2016

1. Let M be a connected Riemannian manifold and $d : M \times M \to \mathbb{R}$ the metric defined by the Riemannian metric tensor as in Theorem 4.24. Fix $p \in M$ and $L > 1$. Let $\varphi : U \to \varphi U \subset \mathbb{R}^n$ be a normal chart at p. Prove that there exists $r > 0$ and a normal ball $B = \exp_p\big(B(0,r)\big) \subset U$ such that $\varphi|B : B \to \varphi B$ is L-bilipschitz from the metric space (B, d) to φB, i.e.
$$\frac{1}{L} d(x,y) \leq |\varphi(x) - \varphi(y)| \leq L d(x,y)$$
for every $x, y \in B$.

2. Prove Lemma 4.34: Let $\gamma : [a,b] \to M$ be admissible and V a continuous piecewise smooth vector field along γ. Then there exists Γ, a variation of γ, such that V is the variation field of Γ. If V is proper, Γ can be taken to be proper as well.

3. Let $B(p,r) = \exp_p B(0,r)$ be a normal ball such that $\partial B(p,r)$ is a normal sphere ($= \exp_p \partial B(0,r)$). Prove that, for every $q \in M \setminus B(p,r)$, there exists a point $q' \in \partial B(p,r)$ such that $d(p,q) = r + d(q',q)$.

4. Generalize the first variation formula (Theorem 4.35) to the case of a variation that is not necessarily proper.

5. Suppose that N and N' are submanifolds of M and that $\gamma : [0,d] \to M$ is a unit speed geodesic such that $\gamma(0) \in N$, $\gamma(d) \in N'$, and that $\ell(\gamma) = d = d(N, N') > 0$. Here $d(N,N') = \inf\{d(x,y) : x \in N, y \in N'\}$. (In other words, γ minimizes the distance between N and N'.)
   Show that $\dot\gamma_0 \perp T_{\gamma(0)}N$ and $\dot\gamma_d \perp T_{\gamma(d)}N'$.