Department of Mathematics and Statistics
Riemannian geometry
Exercise 4
16.2.2016

1. Let $A_{ij} \colon \mathbb{R}^m \to \mathbb{R},\ i,j = 1,\dots,n$, be smooth mappings and denote $A = (A_{ij})$. Prove that in the open set $\{x \in \mathbb{R}^m \colon \det A > 0\}$ we have
$$\frac{\partial}{\partial x^k} \log\det A = \operatorname{tr}\left(\frac{\partial A}{\partial x^k} A^{-1}\right)$$
for all $k = 1,\dots,m$. [Recall that the trace of an $n \times n$ matrix (a_{ij}) is the sum of the diagonal entries $\sum_{i=1}^n a_{ii}$.]

2. Let M be a Riemannian manifold and let $U \subset M$ be an open set. The divergence of a vector field $X \in \mathcal{T}(U)$, denoted by $\operatorname{div} X$, is the trace of the linear map $Y \mapsto \nabla_Y X$. Thus $\operatorname{div} X : U \to \mathbb{R}$,
$$(\operatorname{div} X)(p) = \operatorname{tr}(v \mapsto \nabla_v X), \quad v \in T_pM.$$
Suppose that $(U, x),\ x = (x^1,\dots,x^n)$, is a chart. Express $\operatorname{div} X$ in local coordinates.

3. Prove the claims (a), (d), and (e) in Lemma 4.19. That is: Suppose (U, φ) is a normal chart at p. Show that
   (a) If $v = v^i e_i \in T_pM$, then the normal coordinates of $\gamma^v(t)$ are $(tv^1,\dots,tv^n)$ whenever $\gamma^v(t) \in U$.
   (d) If $\varepsilon > 0$ is so small that $\exp_p$ is diffeomorphic in $B(0,\varepsilon) \subset T_pM$, then the set $\{x \in U \colon r(x) < \varepsilon\}$ is the normal ball $\exp_p\big(B(0,\varepsilon)\big)$.
   (e) If $q \in U \setminus \{p\}$, then $\left(\frac{\partial}{\partial r}\right)_q$ is the velocity vector ($= \dot\gamma$) of the unit speed geodesic from p to q (=unique by (a)), and therefore $|\frac{\partial}{\partial r}| \equiv 1$.

4. Prove the claims (c) and (f) in Lemma 4.19. That is: Suppose (U, φ) is a normal chart at p. Show that
   (c) The components of the Riemannian metric (with respect to the normal chart) at p are $g_{ij}(p) = \delta_{ij}$.
   (f) $\partial_k g_{ij}(p) = 0$ and $\Gamma^k_{ij}(p) = 0$.

5. Complete the proof of Theorem 4.24: Suppose that M is a connected Riemannian manifold and define the function $d : M \times M \to \mathbb{R}$ by
$$d(p,q) = \inf_\gamma \ell(\gamma),$$
where the infimum is taken over all admissible paths from p to q. Show that
   (a) $d(p,q) < \infty$ for all $p, q \in M$,
   (b) $d(p,q) \le d(p,z) + d(z,q)$ for all $p, q, z \in M$,
   (c) the metric space topology = the manifold topology.