

Department of Mathematics and Statistics
 Riemannian geometry
 Exercise 3
 9.2.2016

1. Let $\gamma: I \rightarrow M$ be a C^∞ -path. For $t_0, t \in I$, define a mapping (linear isomorphism) $P_{t_0, t}: T_{\gamma(t_0)}M \rightarrow T_{\gamma(t)}M$ by $P_{t_0, t}v = V(t)$, where $V \in \mathcal{T}(\gamma)$ is the parallel transport of $v \in T_{\gamma(t_0)}M$ along γ . Prove that

$$D_t W(t_0) = \lim_{t \rightarrow t_0} \frac{P_{t_0, t}^{-1}W(t) - W(t_0)}{t - t_0}$$

for $W \in \mathcal{T}(\gamma)$. [Hint: Use a parallel frame along γ .]

2. Let M be a Riemannian manifold, $\langle \cdot, \cdot \rangle$ the Riemannian metric, and ∇ the Riemannian connection of M . The Hessian of a real-valued function $u \in C^\infty(M)$ is a 2-covariant tensor field $\text{Hess } f \in \mathcal{T}^2(M)$ defined by

$$\text{Hess } f(X, Y) = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{T}(M).$$

Prove that $\text{Hess } f$ is symmetric and

$$\text{Hess } f(X, Y) = X(Yf) - (\nabla_X Y)f.$$

3. (a) Prove that the mapping $L: \mathcal{T}(M) \times \mathcal{T}(M) \rightarrow \mathcal{T}(M)$,

$$L(X, Y) = L_X Y$$

(= the Lie derivative of Y with respect to X) is not a connection.

- (b) Prove that there exist smooth vector fields $V \in \mathcal{T}(\mathbb{R}^2)$ and $W \in \mathcal{T}(\mathbb{R}^2)$ such that $V = W = \frac{\partial}{\partial x}$ along the x -axis, but the Lie derivatives $L_V(\frac{\partial}{\partial y})$ and $L_W(\frac{\partial}{\partial y})$ are not equal on the x -axis. (Conclusion?)

4. Let $Y \in \mathcal{T}(M)$ be a vector field on a Riemannian manifold M such that $|Y|$ is constant. Prove that $\nabla_X Y$ is always perpendicular to Y (i.e. $\langle \nabla_X Y, Y \rangle = 0$) for all $X \in \mathcal{T}(M)$.

5. The upper half plane model of the hyperbolic plane \mathbb{H}^2 is the manifold $\{(x, y) \in \mathbb{R}^2: y > 0\}$ with the Riemannian metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

Prove that the Christoffel symbols (of the Riemannian connection of \mathbb{H}^2) with respect to $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$ are

$$\Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = -1/y, \quad \Gamma_{11}^2 = 1/y,$$

and all other $\Gamma_{ij}^k = 0$.