Department of Mathematics and Statistics
Riemannian geometry
Exercise 2
2.2.2016

1. Let (U, x), $x = (x^1, \dots, x^n)$, be a chart and let $\partial_1, \dots, \partial_n$ be the associated coordinate frame. Verify that the formula (3.11)
$$\nabla_X Y = \big(a^i b^j \Gamma^k_{ij} + X b^k\big)\partial_k,$$
where $X = a^i\partial_i \in \mathcal{T}(U)$ and $Y = b^i\partial_i \in \mathcal{T}(U)$, defines an affine connection in U.

2. Let $\gamma\colon I \to \mathbb{R}^n$ be a C^∞-path. Show that a vector field $V \in \mathcal{T}(\gamma)$, $V = (V^1, \dots, V^n)$, is parallel with respect to the Euclidean connection if and only if its components V^i are constant functions.

3. Prove that an affine connection ∇ is symmetric if and only if its Christoffel symbols with respect to any coordinate frame are symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$.

4. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, ∇ an affine connection on M, (U, x), $x = (x^1, \dots, x^n)$, a chart, and $\partial_1, \dots, \partial_n$ the associated coordinate frame. Let $\gamma\colon I \to U$ be a C^∞-path, and let $D_t\colon \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ be the covariant differentiation given by Theorem 3.7. Suppose that
$$\langle\partial_i, \partial_j\rangle' = \langle D_t\partial_i, \partial_j\rangle + \langle\partial_i, D_t\partial_j\rangle$$
for every $i, j \in \{1, \dots, n\}$. Prove that
$$\langle V, W\rangle' = \langle D_t V, W\rangle + \langle V, D_t W\rangle$$
for every $V, W \in \mathcal{T}(\gamma)$.

5. Consider on $\mathbb{R}^2$ the connection defined by $\Gamma^1_{11} = x^1$, $\Gamma^1_{12} = 1$, $\Gamma^2_{22} = 2x^2$, the other Christoffel symbols vanishing. (Here x^2 refers to the coordinate function of the chart (x^1, x^2).) Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path $\gamma(t) = (t, 0)$. Compute the parallel transport along γ of the vector $(\partial_2)_0 \in T_0\mathbb{R}^2$.