Department of Mathematics and Statistics
Riemannian geometry
Exercise 10
26.4.2016

1. Let (U, x) be a chart on a Riemannian manifold and let $f \in C^\infty(U)$ be a smooth real valued function. Compute $\operatorname{Hess} f$ in local coordinates and verify that $\operatorname{Hess} f$ is symmetric.

2. Let f be a smooth real valued function on a Riemannian manifold. Prove that
$$\Delta f = \operatorname{div}(\nabla f) = \operatorname{tr} \operatorname{Hess} f$$
with respect to the Riemannian metric.

3. Let M be a Riemannian manifold. Suppose that $f\colon M \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ are smooth. Show that
$$\operatorname{Hess}(h \circ f) = (h'' \circ f) df \otimes df + (h' \circ f) \operatorname{Hess} f.$$

4. Prove Lemma 8.22: Let M be an oriented Riemannian manifold, ω_M its Riemannian volume form, and $V \in \mathcal{T}(M)$. Then the divergence of V,
$$\operatorname{div} V = \operatorname{tr}(X \mapsto \nabla_X V)$$
satisfies
$$L_V \omega_M = (\operatorname{div} V)\omega_M.$$
[Recall "Cartan's magic formula": $L_V \alpha = d i_V \alpha + i_V d\alpha$.]

5. Let M be a Riemannian n-manifold, $p \in M$, and $r(x) = d(x, p)$. Prove that
$$\Delta r(x) = \frac{n-1}{r(x)} + O\big(r(x)\big)$$
as $x \to p$.