I Quasisymmetric Maps

Before going to qc maps & PDE's want to get a picture of global geometric behaviour of qc maps; this is described by the notion of quasisymmetry.

Question: How to quantify (essential) scale invariance? (as in the picture of previous page)

Note: f a similarity $\Longleftrightarrow$ $f(z) = \alpha z + \beta$

$\alpha = re^{i\theta} \in \mathbb{C}$ (rotation + scaling)

$\beta \in \mathbb{C}$ (translation)

$$\Longleftrightarrow \quad \frac{f(z)-f(w)}{f(z)-f(\xi)} = \frac{z-w}{z-\xi} \quad \forall z, w, \xi \text{ (distinct)}$$

(exercise)

1.1. Definition (Tukia–Väisälä) Let $\eta : [0,\infty) \to [0,\infty)$ be continuous & strictly increasing with $\eta(0)=0$, $\eta(t)\to\infty$ as $t\to\infty$.

A map $f : A \to \mathbb{C}$ ($A \subset \mathbb{C}$) is η-quasisymmetric if

$$(1) \qquad \frac{|f(z)-f(w)|}{|f(z)-f(\xi)|} \le \eta\left(\frac{|z-w|}{|z-\xi|}\right) \quad \forall \text{ (distinct) } z, w, \xi \in A.$$

1.2. Remark If $\eta_0(t) \equiv t$, $t \in \mathbb{R}_+$ then

$$f \ \eta_0\text{-quasisymmetric} \Longleftrightarrow f \text{ similarity}$$

Thus "distance" of η from η_0 describes how far f is from being a similarity.

1.3. Lemma. If f is η-quasisymmetric (and $\#A \geq 3$) then

a) $f : A \to f(A)$ is a homeomorphism.

b) $f^{-1} : f(A) \to A$ is $\tilde{\eta}$-quasisymmetric, where
$$\tilde{\eta}(t) = 1/\eta^{-1}(1/t)$$
(Note: $\tilde{\eta}$ cont., increasing, $\tilde{\eta}(0)=0$ & $\tilde{\eta}(\infty)=\infty$ whenever η satisfies these)

c) If g is η_1-quasisymmetric on $f(A)$, then $g \circ f$ is $(\eta_1 \circ \eta)$-quasisymmetric on A.

Proof: a) f is injective since for $z \neq \zeta$ (1) implies
$$\frac{|f(z) - f(w)|}{|f(z) - f(\zeta)|} \leq \eta\left(\frac{|z-w|}{|z-\zeta|}\right) < \infty \implies f(z) \neq f(\zeta)$$
$\leadsto$ injectivity. Further fixing z, ζ gives $f(w) \to f(z)$ when $w \to z$, so that continuity holds.

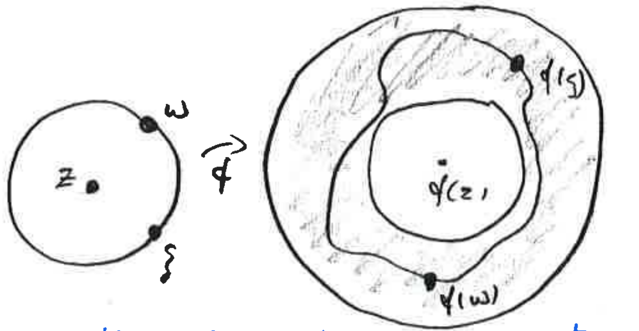
Continuity of f^{-1} follows from b).

b) If $z = f^{-1}a$, $w = f^{-1}b$ and $\zeta = f^{-1}c$, then
$$(1) \iff \frac{|a-b|}{|a-c|} \leq \eta\left(\frac{|z-w|}{|z-\zeta|}\right) \iff \frac{|z-\zeta|}{|z-w|} \leq \frac{1}{\eta^{-1}\left(\frac{|a-b|}{|a-c|}\right)}$$
$$\iff \frac{|f^{-1}a - f^{-1}c|}{|f^{-1}a - f^{-1}b|} \leq \tilde{\eta}\left(\frac{|a-c|}{|a-b|}\right)$$

c) Clear from definition. $\square$

Remarks. As the above proof indicates one can think η as a modulus of continuity and (1) as a similarity invariant form of this.

If $|w-z| = |\xi - z|$, then $|f(w) - f(z)| \le \eta(1)\,|f(\xi) - f(z)|$.



Thus η-quasisymmetric maps map round objects to roundish ones, and "preserve form up to η".

- the notion of quasisymmetry makes sense in a general metric space with $|x-y| = d(x,y)$

1.4. Examples a) If f is L-bilipschitz, i.e.

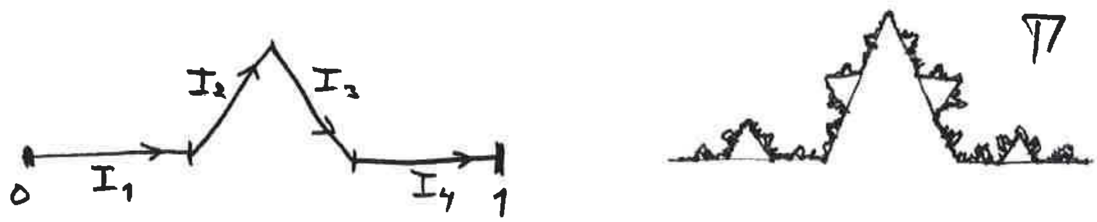
$\frac{1}{L}|x-y| \le |f(x)-f(y)| \le L|x-y| \quad \forall\, x,y \in A$, then f η-q-symm.

where $\eta(t) = L^2 t$ (check!)

b) $f(z) = z|z|^{\alpha-1} = \frac{z}{|z|}|z|^{\alpha}$ is quasisymmetric

(An elementary proof possible, but an easier argument available later in discussing quasiconf. mappings)

c) Consider the snowflake curve Γ:

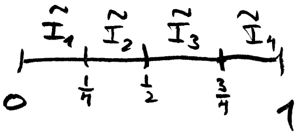


Consider segments I_j, $j=1,\dots,4$, all of length $1/3$ as in left picture.

Let φ_j be the similarity that maps $[0,1]$ onto I_j, preserving orientation as in the picture. Then snowflake Γ is the unique compact set with

$$\Gamma = \bigcup_{j=1}^{4} \varphi_j(\Gamma).$$

There is a similar structure on $[0,1]$ itself:



If $\tilde I_1 = [0,\frac14]$, $\tilde I_2 = [\frac14,\frac12]$, $\tilde I_3 = [\frac12,\frac34]$, $\tilde I_4 = [\frac34,1]$,

and $\tilde\varphi_j[0,1] = \tilde I_j$ [with $\tilde\varphi_j(0) < \tilde\varphi_j(1)$], then $[0,1]$ unique compact with

$$[0,1] = \bigcup_{j=1}^{4} \tilde\varphi_j[0,1].$$

- There is a <u>natural</u> homeo $F: [0,1] \to \Gamma$ with

$$(2) \qquad F(\tilde\varphi_j z) = \varphi_j F(z), \quad z \in [0,1], \quad j = 1,..,4.$$

(For instance, if $\tilde\varphi_{j_1} \circ \dots \circ \tilde\varphi_{j_n}(p) = p$, i.e. p the unique fix point of $\tilde\varphi_{j_1} \circ \dots \circ \tilde\varphi_{j_n}$, let

$$F(p) = q \quad \text{where } \varphi_{j_1} \circ \dots \circ \varphi_{j_n}(q) = q \;)$$

Now (2) $\Rightarrow$

$$F\left(\tilde\varphi_{j_1} \circ \dots \circ \tilde\varphi_{j_n}([0,1])\right) = \varphi_{j_1} \circ \dots \circ \varphi_{j_n}(\underbrace{\Gamma}_{=F[0,1]})$$

where

- $$\operatorname{dia}\left(\tilde\varphi_{j_1} \circ \dots \circ \tilde\varphi_{j_n}[0,1]\right) = \frac{1}{4^n}, \quad \operatorname{dia}\left(\varphi_{j_1} \circ \dots \circ \varphi_{j_n}(\Gamma)\right) = \frac{1}{3^n}.$$

Since $1/3^n = (1/4^n)^\alpha$ where $\alpha = \frac{\log 3}{\log 4}$ $\left(= \frac{1}{\dim_H(\Gamma)}\right)$

it is not difficult to see that

$$(3) \qquad \frac1C |x-y|^\alpha \le |F(x) - F(y)| \le C|x-y|^\alpha \qquad \forall x,y \in [0,1]$$

(To see this scale $F(x)$, $F(y)$ and x,y until the points are fix distance apart; check the details!). It follows that

$$\frac{|F(x) - F(y)|}{|F(x) - F(z)|} \le C^2 \frac{|x-y|^\alpha}{|x-z|^\alpha} = \eta\left(\frac{|x-y|}{|x-z|}\right), \quad \eta(t) = C^2 t^\alpha$$

$\rightsquigarrow F$ η-qsymm.

Note The "Snowflake map" $F : [0,1] \to \Gamma$ of the previous example is not differentiable at any $x \in [0,1]$.

Thus quasisymmetric maps can have a large set of singularities. But how large?

In order to connect the global picture of quasisymmetry to the infinitesimal notions & PDE's, we need to answer the fundamental:

1.5. QUESTION If $f : \Omega \to \Omega'$ quasisymmetric in a domain $\Omega \subset \mathbb{R}^2$, is then f differentiable in a.e. $x \in \Omega$?

Another basic question is how to see weather a map is quasisymmetric? That is:

1.6. QUESTION How to obtain quasisymmetry from local conditions, involving the derivatives only?

These two questions are basic to (any deeper) understanding of qsymmetry. We start with the latter question and study first in which sense conformal mappings are quasisymmetric.

II. Background from Complex Analysis

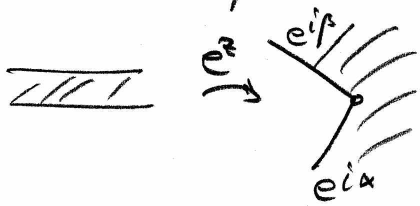
II.1. Conformal mappings

Quasiconformal mappings a generalization of conformal maps, in particular of the geometric aspects of conf. maps.

Hence we briefly recall some basics from geometric function theory.

A map $f: \Omega \to \mathbb{C}$ is conformal, if f is analytic and injective (and $\Omega \subset \mathbb{C}$ domain)

Example: For $0 < \beta - \alpha < 2\pi$, $f(z) = e^z$ is conformal in the strip $\{z : \operatorname{Im} z \in (\alpha, \beta)\}$



2.1. Remark From complex analysis we know:

If $f: \Omega \to \mathbb{C}$ analytic and $z_0 \in \Omega$, then

(i) $f'(z_0) \neq 0 \iff f$ conformal in nbhood of z_0.

(ii) If $f'(z_0) = 0$, then for some $1 < m \in \mathbb{N}$,

$$f(z) = f(z_0) + [\varphi(z)]^m, \quad |z - z_0| < \varepsilon,$$

where φ conformal in $B(z_0, \varepsilon)$ & $\varphi(z_0) = 0$.

In particular, analytic maps are open.

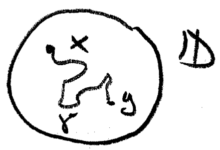
Example $f(z) = z^2$ conformal in $\mathbb{H}_+ = \{z : \operatorname{Re} z > 0\}$, but ~~not~~ in a domain Ω with $0 \in \Omega$.

Remark. A conformal map preserves angles infinitesimally that is, $Df(z)h = f'(z)h$ is a similarity.

II.2 Poincaré Metric

2.2. Definition. For $x, y \in \mathbb{D} = \{z : |z| < 1\}$ let

$$\rho_{\mathbb{D}}(x,y) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1-|z|^2}$$



where "inf" is taken over rectifiable curves γ connecting x to y in $\mathbb{D}$.

2.3. Remark. a) $\varphi : \mathbb{D} \to \mathbb{D}$ Möbius transform $\iff \varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$, where $\alpha \in \mathbb{D}$. $[\varphi(\alpha) = 0]$ For this map φ, derivation $\Rightarrow$

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \equiv \frac{1}{1-|z|^2} \Rightarrow \rho_{\mathbb{D}}(\varphi(z), \varphi(w)) = \rho_{\mathbb{D}}(z,w) \quad \forall z, w \in \mathbb{D}$$

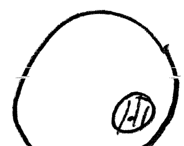
b) $$\rho_{\mathbb{D}}(0,z) = \int_0^{|z|} \frac{2}{1-t^2}\,dt = \log\frac{1+|z|}{1-|z|},$$

and since $\varphi : \mathbb{D} \to \mathbb{D}$ is an isometry in this metric,

c) $$\rho_{\mathbb{D}}(z,w) = \log\left(\frac{1 + \frac{|z-w|}{|1-\bar{w}z|}}{1 - \frac{|z-w|}{|1-\bar{w}z|}}\right) \qquad \left(= \rho_{\mathbb{D}}(\varphi(z), \varphi(w))\right)$$

d) $$|z-w| \leq \tfrac{1}{2}(1-|z|) \Rightarrow \rho_{\mathbb{D}}(z,w) \leq \log\left(\frac{1+1/2}{1-1/2}\right) = \log 3$$

$$\rho_{\mathbb{D}}(z,w) \leq M \Rightarrow |z-w| < \delta(1-|z|), \quad \delta = \delta(M) < 1.$$



II. 3. Koebe Distortion Theorem

Koebe distortion describes general bounds valid for all conformal mappings, and from these one quickly obtains quasisymmetry in Poincaré disks of bounded radius.

As a starting point we have the so called "area - theorem" which will be useful later on.

2.4. Theorem. If f is conformal in $\{|z|>1\}$ and

$f(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, $|z|>1$, then

$$0 \le \frac{1}{2i} \int_{|z|=R} \overline{f(z)}\, f'(z)\, dz = \pi \left(R^2 - \sum_{n=1}^{\infty} \frac{n|b_n|^2}{R^{2n}} \right), \quad R>1.$$

In particular, we have $\sum_{n=1}^{\infty} n|b_n|^2 \le 1$.

For the proof we need Green's formula in complex notation. Recall also the derivatives $\bar\partial = \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$; $\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$

2.5. Lemma (Green's formula) If $F \in C^1(\bar\Omega)$ where $\partial\Omega$ is a C^1-Jordan curve, then

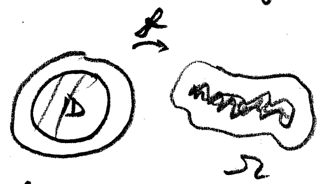
$$\frac{1}{2i} \int_{\partial\Omega} F(z)\, dz = \int_{\Omega} \bar\partial F\, dm$$

Proof: For $F = u + iv$; $LHS = \frac{1}{2i} \int_{\partial\Omega} (u+iv)(dx+idy) =$

$$\frac{1}{2i}\Big(\int_{\partial\Omega} u\,dx - v\,dy\Big) + \frac{1}{2}\int_{\partial\Omega} v\,dx + u\,dy \underset{\uparrow\,\text{Green}}{=} \frac{-i}{2}\int_{\Omega}(-\partial_x v - \partial_y u)\,dm \; +$$

$$+ \frac{1}{2}\int_{\Omega}(\partial_x u - \partial_y v)\,dm \equiv RHS. \qquad \square$$

Proof of Theorem 2.4.: Let us prove first that the integral is positive: If $\Omega = \mathbb{C}\setminus f(|z|>R)$,



$$0 \le |\Omega| = \int_{\Omega} \bar\partial\,\bar w\,dm \underset{\uparrow\,\text{Le. 2.5}}{=} \frac{1}{2i}\int_{\partial\Omega}\bar w\,dw = \frac{1}{2i}\int_{|z|=R}\overline{f(z)}\,f'(z)\,dz$$

$\theta \mapsto f(e^{i\theta})$ parametrizes $\partial\Omega$.

To show the remaining identity plug $f(z) = z + \sum_1^\infty b_n/z^n$ into the integral =>

$$\frac{1}{2i}\int_{|z|=R}\overline{f(z)}\,f'(z)\,dz = \frac{1}{2i}\int_{|z|=R}\Big(\bar z + \sum_1^\infty \frac{\bar b_n}{\bar z^n}\Big)\Big(1 - \sum_{k=1}^\infty \frac{k b_k}{z^{k+1}}\Big)\,dz$$

$$= \frac{1}{2i}\int_{|z|=R}\big(R^2 - |b_1|^2R^{-2} - 2|b_2|^2R^{-4} - \dots\big)\frac{dz}{z} \overset{\text{Cauchy}}{=} \pi\Big(R^2 - \sum_1^\infty \frac{n|b_n|^2}{R^{2n}}\Big),$$

since $\int_{|z|=R} \bar z^{-n} z^{-k}\frac{dz}{z} = 0$ for $n \ne k$. $\square$

With area theorem one can control the coefficients b_n. If $\varphi : \mathbb{D} \to \mathbb{C}$ is conformal in the unit disk $\mathbb{D}$, normalized by $\varphi(0)=0$, $\varphi'(0)=1$, one can apply Thm. 2.4 to

$$f(z) = \frac{1}{\varphi(\frac{1}{z}) - w} \quad , \; w \notin \varphi(\mathbb{D}) \, , \quad \text{and obtain bounds}$$

for coefficients of $\varphi(z)$: E.g. if $\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_2| \le 2$; i.e. $|\varphi''(0)| \le 4$. Combining φ with Möbius transforms one gets bounds for derivatives at arbitrary points $z \in \mathbb{D}$:

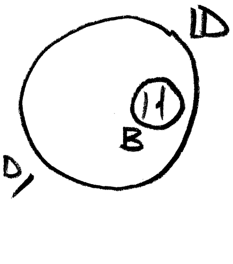
In the enclosed copies from [A-I-M] (next 4 pages) we describe how $|\varphi''(0)| \le 4$ is obtained and how the bounds give following invariant forms of Koebe distortion theorem: [Thm. 2.10.1 in [A-I-M] is our Thm. 2.4]

<u>2.6. Theorem</u> (Thm. 2.10.8 in [A-I-M]) If f is conformal in $\mathbb{D}$ and $z, w \in \mathbb{D}$, then

$$e^{-3\rho_{\mathbb{D}}(z,w)} \le \frac{|f'(z)|}{|f'(w)|} \le e^{3\rho_{\mathbb{D}}(z,w)}$$

<u>Note:</u> By Thm. 2.6, derivatives of conformal maps are "almost" constant in disks

(2.1) $\quad B = \{ z \in \mathbb{D} : |z-w| < \frac{1}{2}(1-|w|) \}$



Integrating the bounds, see [A-I-M]/enclosed page 13.D, we have

<u>2.7. Theorem</u> If f conformal in $\mathbb{D}$ and $z_1, z_2, w \in \mathbb{D}$ with

$$\rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \le M < \infty \quad , \text{ then}$$

$$\frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \le e^{4M} \frac{|z_1 - w|}{|z_2 - w|} \, .$$

e.g. f quasisymm. in Poincaré disks such as (2.1) !

13.A

2.10.2 Koebe $\frac{1}{4}$-Theorem and Distortion Theorem

The Koebe $\frac{1}{4}$-Theorem is one of the first and also one of the most powerful distortion theorems one meets in complex analysis. With the correct interpretation this result implies universal distortion estimates in hyperbolic disks, satisfied by *all* conformal mappings.

To find these we first consider the analytic function $g(z) = z + b_0 + b_1 z^{-1} + \cdots$, which we suppose is conformal in the exterior of the unit disk and further that $g(z) \neq w$ for $|z| > 1$. Then the branch

$$h(z) = \sqrt{g(z^2) - w} = z + \frac{1}{2}(b_0 - w)z^{-1} + \cdots$$

is well defined and conformal in the exterior disk. Furthermore, for any $r > 1$ its restriction to $\{z : |z| > r\}$ extends to a global mapping $h \in W^{1,2}_{loc}(\mathbb{C})$. Thus Theorem 2.10.1 gives

$$|w - b_0| \leqslant 2 \tag{2.67}$$

Often it is convenient to use this result in the following form.

Theorem 2.10.4. *Suppose $g : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, which is conformal in the exterior of the unit disk. If g has the development $g(z) = z + b_0 + b_1 z^{-1} + \cdots$ for $|z| > 1$, then*

$$g(\mathbb{D}) \subset \mathbb{D}(b_0, 2)$$

The famous Koebe $\frac{1}{4}$-theorem is a quick consequence.

Theorem 2.10.5. *Suppose that $\varphi : \mathbb{D} \to \mathbb{C}$ is conformal and normalized by $\varphi(0) = 0$ and $\varphi'(0) = 1$. Then*

$$\mathbb{D}(0, \frac{1}{4}) \subset \varphi(\mathbb{D})$$

Proof. With our assumptions $\varphi(z) = z + a_2 z^2 + \cdots$ for $z \in \mathbb{D}$. The conjugate

$$g(z) = \frac{1}{\varphi(z^{-1})} = z - a_2 + \mathcal{O}(\frac{1}{z}), \qquad |z| > 1,$$

never vanishes, and thus (2.67) implies the classical bound of Bieberbach,

$$|a_2| \leqslant 2 \tag{2.68}$$

Also if $w \notin \varphi(\mathbb{D})$, the function

$$\varphi_1(z) = \frac{w\,\varphi(z)}{w - \varphi(z)} = z + (a_2 + \frac{1}{w})z^2 + \mathcal{O}(z^2)$$

satisfies the assumptions of the theorem, and hence we have additionally

$$|a_2 + \frac{1}{w}| \leqslant 2 \tag{2.69}$$

13.8

Combining the bounds shows that $|w| \geqslant 1/4$ whenever $w \notin \varphi(\mathbb{D})$. $\square$

We may view Theorem 2.10.4 as the counterpart to Koebe's result at ∞. In bounded domains the following form of Koebe's $\frac{1}{4}$-theorem applies in fact to all conformal mappings, independently of their normalization.

Theorem 2.10.6. *Suppose that f is conformal in a domain Ω with $f(\Omega) = \Omega' \subset \mathbb{C}$. Let $z_0 \in \Omega$. Then*

4
$$\frac{1}{4}|f'(z_0)|\operatorname{dist}(z_0, \partial\Omega) \leqslant \operatorname{dist}(f(z_0), \partial\Omega') \leqslant |f'(z_0)|\operatorname{dist}(z_0, \partial\Omega) \qquad (2.70)$$

The first inequality in (2.70) follows from Koebe's theorem, applied to

$$\varphi(z) = \frac{f(z_0 + z\,d) - f(z_0)}{d\,f'(z_0)}, \qquad d = \operatorname{dist}(z_0, \partial\Omega),$$

while the latter inequality is a ~~consequence of the Schwarz lemma~~, applied to $f^{-1} : \mathbb{D}(f(z_0), d') \to \mathbb{D}(z_0, d)$, where $d' = \operatorname{dist}(f(z_0), \partial\Omega')$.

The Bieberbach bound (2.68) also provides us with uniform distortion estimates as soon as we are able to express it in an invariant form. To reveal this we introduce, for each mapping f conformal in $\mathbb{D}$, the *Koebe transform*

$$\varphi(z) = \frac{f(\frac{z+w}{1+\overline{w}z}) - f(w)}{(1-|w|^2)f'(w)}, \qquad z \in \mathbb{D} \qquad (2.71)$$

Here $w \in \mathbb{D}$ is arbitrary.

An elementary calculation gives

$$\varphi''(0) = (1-|w|^2)\frac{f''(w)}{f'(w)} - 2\overline{w} \qquad (2.72)$$

Since φ is conformal in $\mathbb{D}$ with $\varphi(0) = 0$ and $\varphi'(0) = 1$, Bieberbach's coefficient estimate yields the following theorem.

Theorem 2.10.7. *If f is conformal in the unit disk $\mathbb{D}$, then*

$$(1-|w|^2)\frac{|f''(w)|}{|f'(w)|} \leqslant 6, \qquad w \in \mathbb{D}$$

We are now in a position to prove the first of the Koebe distortion theorems. For our purposes an invariant formulation, such as the following, is the most prefered.

Theorem 2.10.8. *Suppose that f is conformal in the unit disk $\mathbb{D}$ and $z, w \in \mathbb{D}$. Then*

$$e^{-3\rho_{\mathbb{D}}(z,w)} \leqslant \frac{|f'(z)|}{|f'(w)|} \leqslant e^{3\rho_{\mathbb{D}}(z,w)}$$

/3.c

Proof. Since f is conformal, the function $g(z) = \log f'(z)$ is analytic in $\mathbb{D}$. Theorem 2.10.7 tells us that $|g'(z)| \leqslant 3\, ds_{hyp}(z)$, and the claim follows via an integration,

$$\left|\log \frac{|f'(z)|}{|f'(w)|}\right| \leqslant |g(z) - g(w)| \leqslant 3\, \rho_{\mathbb{D}}(z,w) \qquad \square$$

It is remarkable that each of Theorems 2.10.5 – 2.10.8 is sharp, as the reader may verify using the Koebe function $f_0(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$. The function f_0 maps the unit disk $\mathbb{D}$ conformally onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

According to Theorem 2.10.8, one may consider the derivative of a conformal mapping as almost constant on hyperbolic disks! It is to be expected that then, with suitable interpretation, the mapping itself should almost be a similarity when restricted to a subdomain bounded in the hyperbolic metric.

This fact turns out to be true and is perhaps most conveniently expressed in the notation of the next theorem. Here note that a homeomorphism f in a domain Ω is a similarity if and only if

$$\frac{|f(z) - f(w)|}{|f(\zeta) - f(w)|} = \frac{|z-w|}{|\zeta - w|} \quad \text{for all } z, w, \zeta \in \Omega \tag{2.73}$$

Theorem 2.10.9. *Suppose that f is conformal in the unit disk $\mathbb{D}$. Let z_1, z_2 and $w \in \mathbb{D}$ with*

$$\rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \leqslant M < \infty$$

Then

$$\frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \leqslant e^{4M} \frac{|z_1 - w|}{|z_2 - w|} \tag{2.74}$$

Proof. We will use the Koebe transform (2.71) and evaluate $\varphi(\zeta_j)$, where $z_j = (\zeta_j + w)/(1 + \overline{w}\zeta_j)$ and $j = 1, 2$. Then $\zeta_j = (z_j - w)/(1 - \overline{w} z_j)$. Hence

$$\frac{f(z_1) - f(w)}{f(z_2) - f(w)} = \frac{\varphi(\zeta_1)}{\varphi(\zeta_2)} = \frac{z_1 - w}{z_2 - w}\, \frac{\varphi(\zeta_1)}{\zeta_1}\, \frac{\zeta_2}{\varphi(\zeta_2)}\, \frac{1 - \overline{w}\, z_2}{1 - \overline{w}\, z_1}$$

To estimate the last expression we note that

$$\begin{aligned} \log\left|\frac{1 - \overline{w}\, z_2}{1 - \overline{w}\, z_1}\right| &\leqslant \log\left(\frac{1 + \left|\frac{z_1 - w}{1 - \overline{w}\, z_1}\right|}{1 - \left|\frac{z_2 - w}{1 - \overline{w}\, z_2}\right|}\right) \\ &\leqslant \rho_{\mathbb{D}}(z_1, w) + \rho_{\mathbb{D}}(z_2, w) \end{aligned}$$

Since $\rho_{\mathbb{D}}(\zeta_j, 0) = \rho_{\mathbb{D}}(z_j, w)$, it remains to show that

$$e^{-3\, \rho_{\mathbb{D}}(\zeta,0)} \leqslant \frac{|\varphi(\zeta)|}{|\zeta|} \leqslant e^{3\, \rho_{\mathbb{D}}(\zeta,0)}, \qquad \zeta \in \mathbb{D} \tag{2.75}$$

C/3. Δ

In fact, by Theorem 2.10.8

$$|\varphi(\zeta)| = \Big|\int_0^\zeta \frac{\varphi'(z)}{\varphi'(0)} dz\Big| \leqslant \int_0^{|\zeta|} e^{3\,\rho_{\mathbb{D}}(z,0)} |dz| \leqslant |\zeta| e^{3\,\rho_{\mathbb{D}}(\zeta,0)}$$

For the former of the inequalities in (2.75), note that this is clear if $|\varphi(\zeta)| \geqslant 1/4$. Otherwise, by Koebe $\frac{1}{4}$-theorem, the interval $[\varphi(\zeta), 0] \subset \varphi(\mathbb{D})$. As $\varphi'(z)dz$ has a constant argument on $\varphi^{-1}[\varphi(\zeta), 0]$, we have

$$|\varphi(\zeta)| = \int_0^{|\zeta|} |\varphi'(z)|\,|dz| \geqslant \int_0^{|\zeta|} e^{-3\,\rho_{\mathbb{D}}(z,0)} |dz| \geqslant |\zeta| e^{-3\,\rho_{\mathbb{D}}(\zeta,0)}$$

Combining these estimates gives the inequality (2.74). □

The above theorem is an invariant version of the second Koebe distortion theorem, expressing in a compact and quantitative manner the fact that locally every conformal mapping is close to a similarity. Here, though, no claim is made on the sharpness of (2.74) in terms of the exponent $4M$. On the other hand, an important fact in Theorem 2.10.9 is the conformal invariance; via a change of variables it applies immediately to all mappings f conformal in a simply connected domain Ω.

We note also the following immediate consequence.

Corollary 2.10.10. *Conformal mappings of the plane are similarities.*

Proof. With a scaling, the estimate (2.74) holds in any disk $\mathbb{D}(0,r) = r\mathbb{D}$. If we denote $M_r = \rho_{r\mathbb{D}}(z_1, w) + \rho_{r\mathbb{D}}(z_1, w)$, then (2.74) attains the form

$$\frac{|z_1 - w|}{|z_2 - w|} e^{-4M_r} \leqslant \frac{|f(z_1) - f(w)|}{|f(z_2) - f(w)|} \leqslant \frac{|z_1 - w|}{|z_2 - w|} e^{4M_r} \tag{2.76}$$

Fixing the points z_1, z_2 and w but letting $r \to \infty$ gives $M_r = \rho_{r\mathbb{D}}(z_1, w) + \rho_{r\mathbb{D}}(z_1, w) = \rho_{\mathbb{D}}(z_1/r, w/r) + \rho_{\mathbb{D}}(z_1/r, w/r) \to 0$. Hence f satisfies (2.73). □

Bounds on the distortion of ratios such as in (2.74) quickly yield a large spectrum of various geometric properties. Indeed, the geometric study of mappings requires general notions that allow such conclusions, and for much larger classes than just (the very rigid) conformal mappings. These considerations will lead in a natural manner to the concept of quasisymmetry, which is studied and utilized in the next section. In this terminology, Theorem 2.10.9 tells us that all conformal mappings are uniformly quasisymmetric in subdomains with bounded hyperbolic diameter.

III. Analytic properties of Quasisymmetric Maps

The goal of this section is to get differentiability properties for quasisymm. maps. (at a.e. points)

III.1. Sobolev Spaces

Intuitively, the Sobolev spaces consist of those functions in $L^p(\Omega)$ which have derivatives contained in $L^p(\Omega)$.

There are several ways to make this rigorous.

3.1. Definition. If $f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^2$ domain, we say that f has weak (or distributional) derivatives $\partial_x f \in L^1_{loc}(\Omega)$, if there is $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f(z)\, \partial_x \varphi(z)\, dm = -\int_\Omega g(z)\, \varphi(z)\, dm \quad \forall \varphi \in C_0^\infty(\Omega)$$

Write $\partial_x f := g$. Similarly for $\partial_y f$.

3.2. Definition. Let $\Omega \subset \mathbb{R}^2$ be a domain and $1 \le p \le \infty$. The Sobolev Space

$$W^{1,p}(\Omega) := \left\{ f \in L^p(\Omega) : f \text{ has weak derivatives } \partial_x f,\ \partial_y f \in L^p(\Omega) \right\}.$$

$W^{1,p}(\Omega)$ has norm $\|f\|_{1,p} := \|f\|_{L^p(\Omega)} + \|\partial_x f\|_{L^p} + \|\partial_y f\|_{L^p}$

which makes it a Banach space.

Example $f(x) = |x| \quad x \in \mathbb{R}$

$f \in W^{1,\infty}(\mathbb{R})$

Another approach to Sobolev Spaces: Recall, that if $F:[a,b] \to \mathbb{C}$ is absolutely continuous [i.e. $\forall \varepsilon > 0$ have $\delta > 0$ s.t. $(x_j, y_j) \subset [a,b]$ disjoint & $\sum_{j=1}^{m} |x_j - y_j| < \delta \Rightarrow \sum_{j=1}^{m} |f(x_j) - f(y_j)| < \varepsilon$] then if this is the case, $F'(x)$ exists for a.e. $x \in (a,b)$ and

$$(3.1) \qquad F(b) - F(a) = \int_a^b F'(t)\,dt$$

[Actually (3.1) characterizes abs. continuity]

Note: If $\psi \in C_0^\infty(a,b) \Rightarrow \psi \cdot F$ is abs. cont., too, and so

$$(3.2) \qquad 0 = \int_a^b (\psi F)'(t)\,dt = \int_a^b \psi(t) F'(t)\,dt + \int_a^b \psi'(t) F(t)\,dt.$$

3.3. Theorem. A function $f \in W^{1,p}(\Omega)$ (domain) $\Leftrightarrow f \in L^p(\Omega)$ and

(i) w.r.t 1-dimensional measure,

$y \mapsto f(x,y)$ abs. cont. for a.e. x,

$x \mapsto f(x,y)$ abs. cont. for a.e. y,

so that partials $\partial_y f(x,y) = \frac{d}{dy} f(x,y)$, $\partial_x f(x,y)$ exist for almost every (x,y), and

(ii) Partials from (i) satisfy $\partial_y f \in L^p(\Omega)$, $\partial_x f \in L^p(\Omega)$.

Proof: "$\Leftarrow$" is easy; use Fubini; from (3.2) above we see that derivatives from abs. continuity are weak derivatives as in Def. 3.1.

"$\Rightarrow$" More difficult; for details see book of Evans & Gariepy: "Measure theory and fine properties of fcns"

$\square$

3.4. Remark Denote

$$W^{1,p}_{loc}(\Omega) = \{ f \in W^{1,p}(U) \ \forall \text{ domain } U, \overline{U} \subset \Omega \}.$$

III.2. Sobolev Properties of Quasisymmetric Maps.

The goal of this section is to show that quasisymmetric maps $f : \Omega \to \Omega'$ (defined) in domains $\Omega, \Omega' \subset \mathbb{R}^2$ are differentiable a.e.; in fact $f \in W^{1,2}_{loc}(\Omega)$. This is non-trivial in view of Example I.1.4. c).

We start with following:

3.5. Definition. If $f \in C(\Omega)$, $\Omega \subset \mathbb{R}^2$ domain, (and $\varepsilon > 0$), the maximal derivative of f is

$$L_f^\varepsilon(z) := \sup \left\{ \frac{|f(z+h) - f(z)|}{|h|} : 0 < |h| < \min\{\varepsilon, \operatorname{dist}(z, \partial\Omega)\} \right\},$$

$z \in \Omega$.

Note: L_f^ε is (Borel)-measurable but quite possibly $L_f^\varepsilon(z) = +\infty$ for some (or even all) $z \in \Omega$.

3.6. Lemma. If $f : \Omega \to \Omega'$ is continuous and $z_0 \in \Omega$, then for all $\varepsilon > 0$,

$$(3.3) \qquad |f(z) - f(z_0)| \leq \int_\gamma L_f^\varepsilon(\zeta)\, |d\zeta| \quad , \quad z \in \Omega,$$

for all rectifiable curves connecting z to z_0 in Ω.

Proof: Can assume γ parametrized by arc length. Then

$$\int_\gamma L_f^\varepsilon(\zeta)\,|d\zeta| = \int_a^b L_f^\varepsilon(\gamma(t))\,|\gamma'(t)|\,dt \quad \text{is well defined}$$

(but possibly $= +\infty$). Fix $\varepsilon > 0$.

1°) Assume $\operatorname{diam}(\gamma) \le \varepsilon$.

Then $\forall\, \xi \in \gamma[a,b]$

$$|f(z) - f(z_0)| \le \frac{|f(z)-f(\zeta)|}{|z-\zeta|}|z-\zeta| + \frac{|f(\zeta)-f(z_0)|}{|z_0-\zeta|}|z_0-\zeta| \le L_f^\varepsilon(\zeta)\,\ell(\gamma).$$

Integrate this along $\gamma \Rightarrow \cancel{\ell(\gamma)}\,|f(z)-f(z_0)| \le \int_\gamma L_f^\varepsilon(\zeta)\,ds \cdot \cancel{\ell(\gamma)}$

2°) If $\operatorname{diam}(\gamma) > \varepsilon$, choose $a = t_0 < t_1 < \dots < t_n = b$ such that $\operatorname{diam}(\gamma[t_j, t_{j+1}]) < \varepsilon$, $0 \le j < n$, and estimate

$$|f(z) - f(z_0)| \le \sum_{j=0}^{n-1} |f(\gamma(t_{j+1})) - f(\gamma(t_j))| \le \sum_{j=0}^{n-1} \int_{\gamma[t_j, t_{j+1}]} L_f^\varepsilon\, ds$$

$$= \int_\gamma L_f^\varepsilon\, ds. \quad \square$$

By the Lemma, L_f^ε is an upper gradient for f, i.e. (3.3) holds $\forall \gamma$.

Ex. If $f \in C^1(\Omega)$ $|\nabla f|$ is an uppergradient.

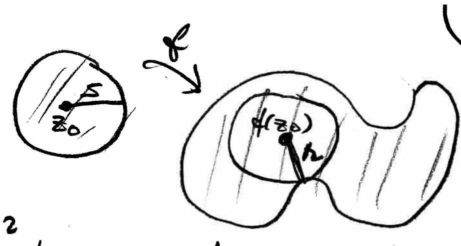
3.7. Lemma If $f : \Omega \to \Omega'$ is η-quasisymmetric, Ω, Ω' domain and $B(z_0, s) \subset \Omega$ disk, then

$$|f(z) - f(z_0)|^2 \le C\,|f B(z_0, s)|, \quad |z - z_0| \le s.$$

Proof: If $r = \min_{|\zeta - z_0| = s} |f(z_0) - f(\zeta)|$, then

$$|f(z) - f(z_0)| \le \eta(1)\, r \quad \text{whenever } |z - z_0| \le s.$$

As the disk $B(f(z_0), r) \subset fB(z_0, s)$



we have
$$|f(z) - f(z_0)|^2 \leq \frac{\eta(1)^2}{\pi} \pi r^2 \leq \frac{\eta(1)^2}{\pi} |fB(z_0, s)| . \quad \square$$

<u>3.8. Lemma</u> If $f : \Omega \to \Omega'$ η-quasisymmetric and $B(z_0, 2r) \subset \Omega$, then $\forall\, 0 < \varepsilon < r$,

$$|\{\xi \in B(z_0, r) : L_f^\varepsilon(\xi) > t\}| \leq c(\eta) \frac{1}{t^2} |fB(z_0, r)| ,$$
$$0 < t < \infty .$$

<u>Remark</u>: By Lemma $L_f^\varepsilon \in$ weak-L^2 in every disk with $2B \subset \Omega$

<u>Proof</u>: [Argument similar as showing maximal fcn is of weak type (1,1).]

Let $E_t = \{z \in B(z_0, r) : L_f^\varepsilon(z) > t\}$, $0 < t < \infty$ fixed.

Then for each $x \in E_t$ can find radius $r_x \in [0, r]$ s.t.

(3.4) $\quad |f(x + r_x e^{i\theta}) - f(x)| > t\, r_x \quad$ (for some θ).

Now
$E_t \subset \bigcup_{x \in E_t} B(x, r_x)$, and using the well known $\frac{1}{5}$-covering lemma [see e.g. Holopainen: "Real Analysis I", notes]

we can find a countable family of <u>disjoint</u> disks $B_j \equiv B(x_j, r_{x_j}) \subset B(z_0, 2r) \subset \Omega$ such that

$$E_t \subset \bigcup_{j=1}^{\infty} 5B_j$$

Then

$$|E_t| \leq \sum_{j=1}^{\infty} |5B_j| = 25\pi \sum_1^{\infty} r_j^2 \leq$$

$$\underset{(3.4)}{\leq} 25\pi \frac{1}{t^2} \sum_{j=1}^{\infty} |f(x_j + r_{x_j} e^{i\theta_j}) - f(x_j)|^2$$

$$\underset{\text{Le. } 3.7}{\leq} 25\pi \frac{1}{t^2} C \sum_1^{\infty} |f B_j| \leq \frac{C}{t^2} |f B(z_0, 2r)|$$

↑ disks disjoint

$$\leq C \frac{1}{t^2} |f B(z_0, r)| . \qquad \square$$

↗ q.symmetry (& exercise)

3.9. Corollary. If f, ε, $B(z_0, r)$ as in Lemma 3.8, then

$$\frac{1}{|B|} \int_B (L_f^\varepsilon)^p \, dm \leq C(\eta, p) \left(\frac{|fB|}{|B|} \right)^{p/2} \qquad 1 \leq p < 2, \quad B = B(z_0, r).$$

Proof: We use the well known identity (based on Fubini)

$$\int_A |g(x)|^p dx = p \int_0^{\infty} t^{p-1} |\{x \in A : |g(x)| > t\}| \, dt ,$$

with $A = B(z_0, r)$, $g = L_f^\varepsilon$. Thus

$$\frac{1}{|B|} \int_B (L_f^\varepsilon)^p \, dm = \frac{p}{|B|} \int_0^{\infty} t^{p-1} |\{x \in B : L_f^\varepsilon(x) > t\}| \, dt$$

$$\leq \frac{p}{|B|} \int_0^{t_0} t^{p-1} |B| \, dt + \frac{p}{|B|} C(\eta) \int_{t_0}^{\infty} t^{p-3} |fB| \, dt$$

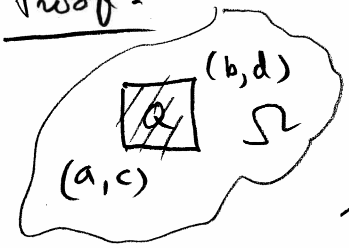
$$\leq t_0^p + \frac{p\, C(\eta)}{2-p} t_0^{p-2} \frac{|fB|}{|B|} \leq c \left(\frac{|fB|}{|B|} \right)^{p/2}$$

when we choose $t_0 = \sqrt{|fB|/|B|}$. $\square$

3.10. Theorem. If $f : \Omega \to \Omega'$ is η-quasisymmetric, $\Omega \subset \mathbb{R}^2$ domain, then

$$f \in W^{1,1}_{loc}(\Omega).$$

<u>Proof:</u> Let $Q := [a,b] \times [c,d] \subset \Omega$,



$\varepsilon < \operatorname{dist}(Q, \partial\Omega)$. By Corollary 3.9 & Fubini ($L_f^\varepsilon \in L^1(\Omega)$)

for a.e. $y \in [c,d]$ the integral

$$x \longmapsto \int_a^x L_f^\varepsilon(t,y)\,dt \quad \text{is abs. cont. on } [c,d]$$

Therefore, whenever $(a_j, b_j) \subset [a,b]$ are disjoint, $j = 1, \dots, n$ we have

$$\sum_{j=1}^n |f(a_j,y) - f(b_j,y)| \underset{\text{Lemma 3.6}}{\leq} \sum_{j=1}^n \int_{a_j}^{b_j} L_f^\varepsilon(t,y)\,dt$$

$$= \int_{\bigcup_1^n (a_j,b_j)} L_f^\varepsilon(t,y)\,dt < \varepsilon, \text{ whenever } \sum_{j=1}^n |a_j - b_j| < \delta = \delta(y).$$

Thus $x \mapsto f(x,y)$ abs. cont for a.e. $y \in [c,d]$.

Similarly, $y \mapsto f(x,y)$ abs. cont. for a.e. $x \in [a,b]$.

Moreover
$$|\partial_x f(x,y)|, \ |\partial_y f(x,y)| \leq L_f^\varepsilon(x,y) \underset{\text{Corollary 3.9}}{\in} L^1(Q)$$

Thus $\partial_x f, \partial_y f \in L^1(\Omega)$ and $f \in W^{1,1}_{loc}(\Omega)$ by Thm. 3.3.

$\square$

<u>Remark.</u> $f = u + iv$ $\mathbb{C}$-valued. If necessary, can study Sobolev properties of u, v separately.

Next, want to show that any qsymm. $f \in W^{1,2}_{loc}(\Omega)$.

For this need some real analysis: Given any (see e.g. Rudin: Real and Complex Analysis) finite positive Borel measure ν on $\mathbb{R}^n$, we have the decomposition: ($A \subset \mathbb{R}^n$ Borel)

(3.5a) $$\nu(A) = \int_A g(x)\,dx + \lambda(A)$$

where

(3.5.b) $$g(x) := \lim_{r\to 0} \frac{\nu(B(x,r))}{|B(x,r)|} \equiv D\nu(x) \ \exists \text{ for a.e. } x \ \& \ g \in L^1(\mathbb{R}$$

and where $\lim_{r\to 0} \frac{\lambda(B(x,r))}{|B(x,r)|} = 0$, i.e. λ singular part of ν, $g(x)dx$ the abs. cont. part of ν.

<u>3.11. Theorem.</u> If $f:\Omega \to \Omega'$ 2-qsymm. and Ω, Ω' domain with $\Omega' = f(\Omega)$ bounded, then $f \in W^{1,2}_{loc}(\Omega)$. In fact, we have $L_f \in L^2(\Omega)$, where

$$L_f(x) := \limsup_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}.$$

<u>Proof:</u> Note that $\varepsilon \mapsto L_f^\varepsilon(x)$ decreasing, and $L_f(x) = \lim_{\varepsilon\to 0} L_f^\varepsilon$.

Lemma 3.7 $\Rightarrow$ If $B(z_0,r) \subset \Omega$,

(3.6) $$\frac{|f(z)-f(z_0)|^2}{|z-z_0|^2} \le C \frac{|f B(z_0,r)|}{|B(z_0,r)|}, \quad |z-z_0| = r$$

Considering the finite measure $\nu(A) = |f(A\cap\Omega)|$, we have a.e

$$|\partial_x f|^2, |\partial_y f|^2 \le L_f(x)^2 \underset{(3.6)}{\le} C \lim_{r\to 0} \frac{\nu(B(x,r))}{|B(x,r)|} = C D\nu(x) \underset{(3.5)}{\in} L^1(\Omega$$

$\square$

Thus $f \in W^{1,2}(\Omega)$ whenever f qsymm. & $f(\Omega)$ bdd.

In general have $f \in W^{1,2}_{loc}(\Omega)$ for any qsymm. (in a domain

We will need even a further refinement:

3.12. Theorem (Gehring-Lehto) If $f: \Omega \to \Omega'$ is a $W^{1,1}_{loc}$-homeomorphism, then $f(x)$ is differentiable at a.e. $x \in \Omega$.

In fact, the result remains true if $f \in W^{1,2}_{loc}(\Omega)$ is an open mapping.

(For the details of the proof see the next 4 pages, copied from [A-I-M].)

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First we establish a fairly well-known and general theorem of Gehring and Lehto [137] which asserts that an open mapping with finite partial derivatives at almost every point is differentiable at almost every point. For homeomorphisms the result was earlier established by Menchoff [258]. The proof must use properties of the plane, as it is false in higher dimensions, although certain analogs exist; see [180]. Second, with the Gehring-Lehto result we are able to connect the volume derivatives and the pointwise Jacobians $J(z,f)$ and thereby to obtain a first version of the change-of-variables formula.

3.3.1 The Differentiability of Open Mappings

We recall that a mapping $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is *open* if $f(U)$ is open for every open $U \subset \Omega$.

To begin with we will need the following refinements of the concept of density. Let E be a measurable subset of $\mathbb{C}$. A point $z_0 = x_0 + iy_0 \in E$ is called a point of x-density if x_0 is a point of linear density of the set $\{x \in \mathbb{R} : x + iy_0 \in E\}$. Similarly we have the notion of y-density. A point $z_0 \in E$ that is both a point of x-density and of y-density will be called a point of xy-density. Of course, as soon as we are able to establish the measurability of the set of points of xy-density, then Fubini's theorem implies that such points have full measure in E. Indeed, the set E_1 consisting of points of x-density has full measure in E for otherwise $|E \setminus E_1| > 0$ and by Fubini's theorem we could find y_0 such that the set $\{x \in \mathbb{R} : x + iy_0 \in E \setminus E_1\}$ has positive linear measure. But this would contradict the Lebesgue density theorem on the real line. Analogously, the set E_2 of points of y-density is measurable with full measure and therefore the intersection of these two sets has full measure. This is of course the set of all points of xy-density.

To show the measurability of E_1, it is enough to consider closed sets E. We denote by $E_{n,k}$ the set of points $x + iy \in E$ such that

$$\mathcal{H}^1(\{t \in [a,b] : t + iy \in E\}) \geqslant (1 - \frac{1}{n})(b-a)$$

whenever

$$a < x < b \quad \text{and} \quad 0 < b - a \leqslant \frac{1}{k}$$

Then clearly

$$E_1 = \bigcap_{n=1}^{\infty} \bigcup_{k}^{\infty} E_{n,k},$$

and it suffices to show that the sets $E_{n,k}$ are closed. Here, let $z_j = x_j + iy_j \in E_{n,k}$ with $z_j \to z_0 = x_0 + iy_0$. If $a < x_0 < b$ with $b - a < 1/k$, then $a < x_j < b$ for all j large enough, and since E is closed,

$$\begin{aligned} \mathcal{H}^1(\{t \in [a,b] : t + iy_0 \in E\}) &\geqslant \mathcal{H}^1(\cap_{\ell=1}^{\infty} \cup_{j=\ell}^{\infty} \{t \in [a,b] : t + iy_j \in E\}) \\ &\geqslant (1 - \frac{1}{n})(b-a) \end{aligned}$$

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Therefore $z_0 \in E_{n,k}$ which proves that $E_{n,k}$ is closed and hence that E_1 is measurable. We argue in the same manner to see E_2 is measurable. We have thus established that the set of all points of xy-density in E is measurable with full measure.

We next have the following lemma whose proof is an elementary argument in linear density and measure theory.

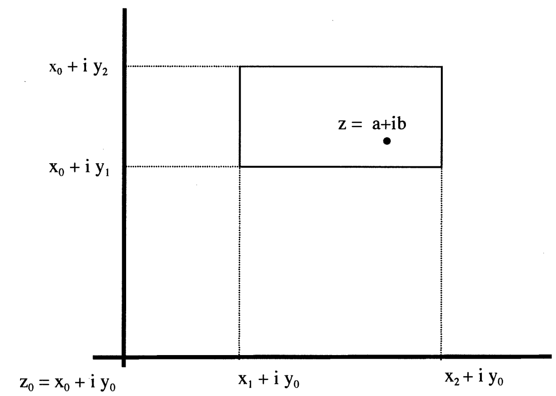
Lemma 3.3.1. *Let $\varepsilon > 0$ and $z_0 = x_0 + iy_0 \in E$ be a point of xy-density of E. Then, for all $z = a + ib$ sufficiently close to z_0, there is a rectangle*

$$R = [x_1, x_2] \times [y_1, y_2]$$

containing z and such that

$$(x_2 - x_1) < 2\,\varepsilon\,|a - x_0|, \quad (y_2 - y_1) < 2\,\varepsilon\,|b - y_0|$$

and such that the points $x_1 + iy_0$, $x_2 + iy_0$, $x_0 + iy_1$ and $x_0 + iy_2$ all lie in E.



Choosing the rectangle $R = [x_1, x_2] \times [y_1, y_2]$

Proof. We may assume that $x_0 = y_0 = 0$, and so z_0 is the point at the origin. We may also assume $a, b > 0$. Let

$$E_x = \{x \in \mathbb{R} : x + i0 \in E\}, \qquad E_y = \{y \in \mathbb{R} : 0 + iy \in E\}$$

Since $x = 0$ is a point of density of E_x, each of the intervals $(a - \varepsilon a, a)$ and $(a, a + \varepsilon a)$ contains points of E_x provided that a is sufficiently small, say $a < \delta$. We pick $x_1 \in (a - \varepsilon a, a)$ and $x_2 \in (a, a + \varepsilon a)$ such that both $x_1 + i0$ and $x_2 + i0$ lie in E. Similarly, we find $y_1 \in (b - \varepsilon b)$ and $y_2 \in (b, b + \varepsilon b)$ with both $0 + iy_1$ and $0 + iy_2 \in E$, again provided b is sufficiently small. Now we have $x_2 - x_1 < 2\varepsilon a$ and $y_2 - y_1 < 2\varepsilon b$, as desired. □

We shall now prove the following theorem.

Theorem 3.3.2. *Let $f : \Omega \to \mathbb{C}$ be a continuous open mapping. Then f is differentiable almost everywhere in Ω if and only if f has finite first partials almost everywhere.*

Proof. If f is differentiable almost everywhere, then f has finite first partials almost everywhere. It is the converse that we need to establish. Thus we assume that the partials f_x and f_y exist and are finite at almost every point of Ω. It will be enough to prove that f is in fact differentiable at almost every point of a given compact subset $X \subset \Omega$. Let t be a real number, $0 < |t| < \operatorname{dist}(X, \partial\Omega)$. We define, for those $z = x + iy$ at which both partials exist and are finite, the function

$$F_t(z) = \left|\frac{f(x+t,y) - f(x,y)}{t} - f_x(x,y)\right| + \left|\frac{f(x,y+t) - f(x,y)}{t} - f_y(x,y)\right| \tag{3.4}$$

It is easy to see that the set where F_t is defined is a Borel set [316, p. 70]. Thus F_t is a Borel function defined almost everywhere on X and it follows that the functions

$$g_n(z) = \sup_{0<|t|<1/n} F_t(z)$$

are also Borel for sufficiently large n (as it suffices to let t run through only rational values). From our assumption on the partial derivatives of f, we see that

$$g_n(z) \to 0, \quad \text{almost everywhere as } n \to \infty$$

Now by the theorems of Egoroff and Lusin, there is an increasing sequence of compact subsets $X_1 \subset X_2 \subset \cdots \subset X$ with

$$|X - \bigcup_{\nu=1}^{\infty} X_\nu| = 0$$

for which we have for each ν that the functions f_x, f_y are continuous in X_ν and

$$F_t(z) \to 0, \quad \text{uniformly on } X_\nu$$

Of course, it will now suffice to fix a ν, put $E = X_\nu$ and prove that f is differentiable at any $z_0 \in E$ that is an xy-density point of E.

Let $0 < \varepsilon < 1$. We aim to prove the estimate

$$\begin{aligned} &|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\ &\quad \leqslant \varepsilon(4 + |f_x(z_0)| + |f_y(z_0)|)(|x - x_0| + |y - y_0|) \end{aligned} \tag{3.5}$$

whenever $z \in \Omega$ is sufficiently close to z_0. This is enough to ensure the differentiability of f at z_0.

Now, for all $z \in E$ sufficiently close to z_0, we have

$$|f_x(z) - f_x(z_0)| < \varepsilon, \qquad |f_y(z) - f_y(z_0)| < \varepsilon, \tag{3.6}$$

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while $|F_t(z)| < \varepsilon$ if t is small. Next

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\
&\quad \leqslant \ |f(z) - f(x + iy_0) - f_y(x + iy_0)(y - y_0)| \\
&\qquad + |f(x + iy_0) - f(z_0) - f_x(z_0)(x - x_0)| \\
&\qquad + |f_y(x + iy_0) - f_y(z_0)||y - y_0|
\end{aligned}$$

We now assume that z is sufficiently close to z_0 so as to be able to apply (3.6). Accordingly, we arrive at the estimate

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(y - y_0)| \\
&\quad \leqslant \ F_{y-y_0}(x + iy_0)|y - y_0| + F_{x-x_0}(z_0)|x - x_0| + \varepsilon|y - y_0| \\
&\quad \leqslant \ 2\varepsilon(|x - x_0| + |y - y_0|) \qquad (3.7)
\end{aligned}$$

whenever $z = x + iy \in \Omega$ is sufficiently close to z_0 and in addition $x + iy_0 \in E$. Similarly, we have this same estimate whenever $z = x + iy \in \Omega$, $x_0 + iy \in E$ and z is sufficiently close to z_0.

Up to this point we have not used the fact that f is open, and we do so now. That f is open implies that f satisfies the maximum principle—maxima occur on the boundary. In particular, for each point $z \in \Omega$ close to z_0, let R be the rectangle given by Lemma 3.3.1. Using the maximum principle, we find that the expression

$$|f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)|$$

considered as a function of $\zeta = u + iv$ takes its maximum value on the boundary of R. Hence at the maximum point $\zeta \in \partial R$,

$$\begin{aligned}
&|f(z) - f(z_0) - f_x(z_0)(x - x_0) - f_y(z_0)(x - y_0)| \\
&\quad \leqslant \ |f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)| \\
&\qquad + |f_x(z_0)||u - x| + |f_y(z_0)||v - y|
\end{aligned}$$

Furthermore, for each boundary point $\zeta = u + iv \in \partial R$, either $u + iy_0 \in E$ or $x_0 + iv \in E$. In view of the estimate in (3.7),

$$|f(\zeta) - f(z_0) - f_x(z_0)(u - x_0) - f_y(z_0)(v - y_0)| \leqslant 2\varepsilon(|u - x_0| + |v - y_0|)$$

As $|u - x| \leqslant \varepsilon|x - x_0|$ and $|v - y| \leqslant \varepsilon|y - y_0|$, the above estimates prove (3.5), establishing the theorem. □

We note that in the proof we used only the maximum principle on rectangles, an apparently weaker condition than assuming f is open.

The above theorem applies, in particular, to Sobolev homeomorphisms.

Corollary 3.3.3. *Every homeomorphism $f \in W^{1,1}_{loc}(\Omega)$ is differentiable almost everywhere.*

Finally, we come to the definition of a K-quasiconformal mapping.

3.13. Definition. A homeomorphism $f:\Omega\to\Omega'$ (domain) is K-quasiconformal, if $f\in W^{1,2}_{loc}(\Omega)$ and

$$\max_\alpha |\partial_\alpha f(x)| \le K \min_\alpha |\partial_\alpha f(x)| \quad \text{a.e. } x\in\Omega.$$

and $J(z,f)\ge 0$ a.e. (orientation preserving)

Remark By Gehring–Lehto theorem a K-qc f is differentiable a.e. in Ω and therefore directional derivatives

$$\partial_\alpha f(x) = \lim_{r\to 0}\frac{f(x+re^{i\alpha})-f(x)}{r} = \lim_{r\to 0}\frac{f(x_1+r\cos\alpha, x_2+r\sin\alpha)-f(x)}{r}$$

$$= Df(x)e^{i\alpha}$$

exist for all $\alpha\in[0,2\pi]$ at a.e. $x\in\Omega$.

Note: For a conformal map, $Df(z)e^{i\theta} = f'(z)e^{i\theta} \Rightarrow |\partial_\alpha f(z)| = |\partial_\beta f(z)|$ $\forall \alpha,\beta$

3.14. Theorem If $f:\Omega\to\Omega'$ is η-quasisymmetric, then f is K-quasiconformal with $K=\eta(1)$.

Proof:
$$\frac{|f(z+re^{i\alpha})-f(z)|}{r} \le \eta(1)\frac{|f(z+re^{i\beta})-f(z)|}{r}$$

which shows, in the limit $r\to 0$, that

$|\partial_\alpha f(x)| \le \eta(1)\,|\partial_\beta f(x)|$ $\forall\alpha,\beta\in[0,2\pi]$. That $f\in W^{1,2}_{loc}$ is shown in Thm 3.11. (and that f homeo in Lemma 1.3.) $\square$

Our next major goal is the converse, that any K-quasiconformal map is quasisymmetric in disks $B(z_0, r)$ where $r < \mathrm{dist}(z_0, \partial\Omega)$; just as for conformal maps in Theorem 2.7.

Before this let us collect few very important (other) properties of quasisymmetric maps.

Lusin's property (N)

3.15. Definition. Let $f: \Omega \to \Omega'$ be a measurable mapping. We say that

- f has Lusin's property (N) if
$$|E| = 0 \Rightarrow |f(E)| = 0, \quad E \subset \Omega.$$

- f has Lusin's property (N^{-1}) if
$$|E| = 0 \Rightarrow |f^{-1}(E)| = 0, \quad E \subset \Omega'.$$

Property (N) is needed e.g. for changing variables with f, and (N^{-1}) to guarantee that $u \circ f$ is measurable for any measurable u on Ω'.

3.16. Lemma If $f : \Omega \to \Omega'$ is η-quasisymmetric, $z_0 \in \Omega$ and $|z - z_0| = r < d(z_0, \partial\Omega)$, then

$$|f(z) - f(z_0)|^2 \le C \int_{B(z_0,r)} (L_f)^2 \, dm$$

Proof: From Theorem 3.11 we know $L_f \in L^2_{loc}(\Omega)$.
May assume $z_0 = 0$; Let $|z - z_0| = r$, if $|w| = r$, quasisymmetry gives us

$$|f(z) - f(0)| \le \eta(1)\, |f(w) - f(0)| \le \eta(1)\eta(2)\, |f(w) - f(w/2)|$$
$$\underset{\text{Lemma 3.6}}{\le} \eta(1)\eta(2) \int_{w/2}^{w} L_f^\varepsilon$$

Integrating over circle $|w| = r$ implies

$$|f(z) - f(0)| \le \frac{C}{r} \int_{\frac{r}{2} < |w| < r} L_f^\varepsilon \le \frac{C}{r} \int_{B(z_0,r)} L_f^\varepsilon$$

Next let $\varepsilon \to 0$; by monotone convergence get

$$|f(z) - f(0)| \le \frac{C}{r} \int_{B(z_0,r)} L_f \, dm \le \frac{C}{\cancel{r}} \sqrt{\int_{B(z_0,r)} L_f^2} \; \cancel{\sqrt{|B(z_0,r)|}}$$

which gives $|f(z_0) - f(0)|^2 \le C \int_{B(z_0,r)} (L_f)^2 \, dm$ $\square$

3.17. Theorem If $f:\Omega\to\Omega'$ is η-quasisymmetric, then

$$|E|=0 \iff |fE|=0 \quad , \quad \forall\, E\subset\Omega \text{ measurable.}$$

Proof: "$\Rightarrow$" enough. May assume $E\subset B=B(z_0,r)$; $r<\frac{1}{4}\operatorname{dist}(z_0,\partial\Omega)$.
If $|E|=0$, cover $E=\bigcup_{j=1}^{\infty} B_j$ by disks $B_j=B(x_j,r_j)$ with

$$\sum_{j=1}^{\infty} \operatorname{dia}(B_j)^2<\delta$$

Using Vitali type covering lemmas (see Real Analysis I) we find a subfamily $\{\tilde B_k\}\subset\{B_j\}$ such that

- $\tilde B_k\cap\tilde B_{k'}=\emptyset$ for $k\neq k'$
- $E\subset\bigcup_k(5\tilde B_k)$

Then Quasisymmetry/Exercises $\Rightarrow$ $|f(5\tilde B_k)|\le C(\eta)\,|f(\tilde B_k)|$.
Thus

$$|fE|\le\sum_k|f(5\tilde B_k)|\le C(\eta)\sum_k|f(\tilde B_k)|\le c(\eta)\sum_k\operatorname{dia}(f\tilde B_k)^2$$

$$\underset{(\text{Le } 3.16)}{\le}\; c\sum_k\int_{\tilde B_k}L_f^2 \underset{(\text{disks disjoint})}{=} C(\eta)\int_{\bigcup_k\tilde B_k}L_f^2 .$$

Since $L_f\in L^2(B)$, $E\mapsto\int_E L_f^2$ is abs. continuous, so that
$|\bigcup_k\tilde B_k|\le\sum\operatorname{dia}(\tilde B_k)^2<\delta\Rightarrow\int_{\bigcup_k\tilde B_k}L_f^2<\varepsilon$. Thus $E\mapsto|fE|$, too, is abs. continuous. $\square$

3.18. Corollary If $f: \Omega \to \Omega'$, $g: \Omega' \to \Omega''$ are quasisymm. (and orientation preserving), then

a) $$|f(E)| = \int_E J(z,f)\, dm = \int_E \det(Df(z)), \quad E \subset \Omega.$$

b) $$D(g\circ f)(x) = Dg(f(x))\, Df(x) \quad \text{for a.e. } x \in \Omega.$$

Proof:
a) Almost everywhere $f(x+h) = f(x) + Df(x)h + h\,\varepsilon(h)$,

so that $\dfrac{|f B(x,r)|}{|B(x,r)|} \to \det(Df(x))$ a.e., and claim

follows from (3.5 a), b)), applied to measure $\nu(A) = |f(A\cap\Omega)|$.

b) Since f & g are differentiable a.e and f preserves sets of measure zero, b) follows from chain rule. $\square$

3.19. Remark If $f \in W^{1,1}_{loc}(\Omega)$ is a homeomorphism, but does not necessarily satisfy Lusin's condition (N), applying (3.5 a) – (3.5 b) gives, for $\nu(A) = |f(A\cap\Omega)|$ & A Borel set,

a) $$\int_E J(x,f)\, dm \leq \underbrace{|fE|}_{(E \text{ Borel})} \qquad (J(x,f) \equiv \det(Df(x)) \text{ exists a.e.})$$

and

b) $$\int_\Omega \phi\circ f(x)\, J(x,f)\, dm \leq \int_{f\Omega} \phi(\xi)\, dm, \quad \phi \in C(\Omega)\cap L^1($$

Proof of b): Note first that since ϕ is assumed to be continuous,

$\phi \circ f$ is continuous. For b) note that if $E \subset \Omega$ is a Borel set and $\phi = \chi_{fE}$, a) $\Leftrightarrow$

$$\int_{\Omega} \chi_{fE} \circ f(x)\, J(x,f) \le \int_{\Omega'} \chi_{fE}(\xi)\, dm$$

By linearity b) holds when $\phi = \sum_1^n a_j \chi_{fE_j}$ is a simple fcn, and since a continuous ϕ is a locally uniform limit of such functions, b) follows.

Global Quasiconformal maps are Quasisymmetric

We want to show that (according to Definition 3.13) if $f: \mathbb{C} \to \mathbb{C}$ is a $W^{1,2}_{loc}$-homeomorphism and for a.e z,

$$(3.7) \qquad \max_{\alpha} |\partial_\alpha f(z)| \le K \min_{\alpha} |\partial_\alpha f(z)|$$

then f is quasisymmetric. Looking at the singular values of $Df(x)$ we see that (3.7) is equivalent to

$$(3.8) \qquad |Df(z)|^2 \le K\, J(z,f), \quad \text{a.e. } z \in \mathbb{C}$$

For proving the quasisymmetry we need a few auxiliary lemmas. In the first, if $u: Y \to \mathbb{R}$ is continuous with Y compact, write

$$\operatorname{osc}_u(Y) = \max_{z,w \in Y} |u(z) - u(w)|$$

3.20 Lemma (Oscillation Lemma) Suppose $u \in W^{1,2}(B(0,r))$ is continuous in $\overline{B(0,r)}$ and real valued. Then for $\varrho < r$,

$$\int_\varrho^r \operatorname{osc}_u(S^1(t))^2 \, \frac{dt}{t} \leq 2\pi \int_{\varrho<|z|<r} |\nabla u|^2$$

Here $S^1(t) = \{ z : |z| = t \}$.

Proof As $u \in W^{1,2}$, u is abs. continuous on almost all circles $S^1(t)$, $\varrho < t < r$, [as $u \circ \exp \in W^{1,2}$ is abs. continuous on lines; see Exercises]. Thus for a.e. $t \in [\varrho, r]$,

$$\operatorname{osc}_u(S^1(t)) \leq \int_0^{2\pi} |\nabla u|(te^{i\theta})\, t\, d\theta \overset{\text{Hölder}}{\leq} t \left[2\pi \int_0^{2\pi} |\nabla u|^2 (te^{i\theta})\, d\theta \right]^{1/2}$$

Squaring gives

$$\int_\varrho^r \operatorname{osc}_u^2(S^1(t))\, \frac{dt}{t} \leq 2\pi \int_\varrho^r t \int_0^{2\pi} |\nabla u|^2 (te^{i\theta})\, d\theta . \qquad \square$$

As a second lemma we need a preliminary version of change of variables. At the moment we need to content with the following, since we do <u>not</u> know <u>yet</u> if a qconf f has Lusin properties.

3.21. Lemma Let $f \in W^{1,2}(\mathbb{C})$ be a quasiconf. homeo. (so that (3.7) holds). Then if $v \in Lip_1(\Omega)$, we have

a) $\quad v \circ f \in W^{1,2}_{loc}(\Omega')$, $\quad \Omega' = f^{-1}(\Omega)$, and

b) $$\int_{\Omega'} |\nabla(v \circ f)|^2 \leq K \int_\Omega |\nabla v|^2$$

Proof: That $v \circ f \in W^{1,2}_{loc}(\Omega)$ follows from Thm. 3.3.

To prove b) assume first that $v \in C^\infty(\Omega)$. Then at points of differentiability of $f(z)$ we have

$$|\nabla(v \circ f)|^2(z) = |Df(z)^t|^2 |\nabla v|(f(z)) \le K |\nabla v|^2(f(z)) J(z,f)$$

and Remark 3.19 b) gives

$$\int_{\Omega'} |\nabla(v \circ f)|^2 \le K \int_{\Omega'} |\nabla v|^2 \circ f \; J(z,f) \le K \int_{\Omega} |\nabla v|^2 < \infty,$$

proving the claim for smooth v. In the general case, when ∇v not continuous, cannot use Remark 3.19; instead note that if $|\nabla v| \in L^2(\Omega)$, there are $v_\varepsilon \in C^\infty(\Omega)$ with $\|\nabla v - \nabla v_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$. So if $\phi \in C_0^\infty(\Omega')$ then

$$\int_{\Omega'} \phi \, \nabla(v \circ f) = -\int_{\Omega'} (v \circ f) \nabla \phi = -\lim_{\varepsilon \to 0} \int_{\Omega'} (v_\varepsilon \circ f) \nabla \phi$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega} \phi \, \nabla(v_\varepsilon \circ f) \le \lim_{\varepsilon \to 0} \|\phi\|_{L^2} \|\nabla(v_\varepsilon \circ f)\|_{L^2(\Omega')}$$

$v_\varepsilon \in C^\infty$

$$\le \sqrt{K} \, \|\phi\|_{L^2(\Omega')} \varlimsup_{\varepsilon \to 0} \|\nabla v_\varepsilon\|_{L^2(\Omega)} = \sqrt{K} \, \|\phi\|_{L^2} \|v\|_{L^2}$$

Taking the supremum over $\phi \in C_0^\infty(\Omega')$ with $\|\phi\|_{L^2} = 1$ proves the claim b). $\square$

Now we are to show that for homeos $f: \mathbb{C} \to \mathbb{C}$

f quasisymmetric $\iff$ f quasiconformal

Let us first fix the following negligence we made in Def. 3.13:

We will always assume that a quasiconformal mapping is orientation preserving ($\Leftrightarrow J(z,f) \geq 0$).

$$\boxed{\begin{array}{l} \text{Thus } f:\Omega \to \Omega' \text{ is a } K\text{-quasiconformal map if} \\ (3.9.a) \quad f \in W^{1,2}_{loc}(\Omega) \;\&\; f \text{ homeo}, \\ (3.9.b) \quad \max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)|, \text{ a.e. } z \in \Omega \\ \text{and } (3.9.c) \quad J(z,f) \geq 0. \end{array}}$$

With this, let us proceed to one of main Theorems in this course:

3.22. Theorem. If $f: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal then f is η-quasisymmetric.

Remarks: • Converse was done in Theorem 3.14

• The proof gives a highly non-optimal η. A better one is given in Corollary 4.5 below.

Proof of Theorem 3.22: We need to find $\eta = \eta_K$ with

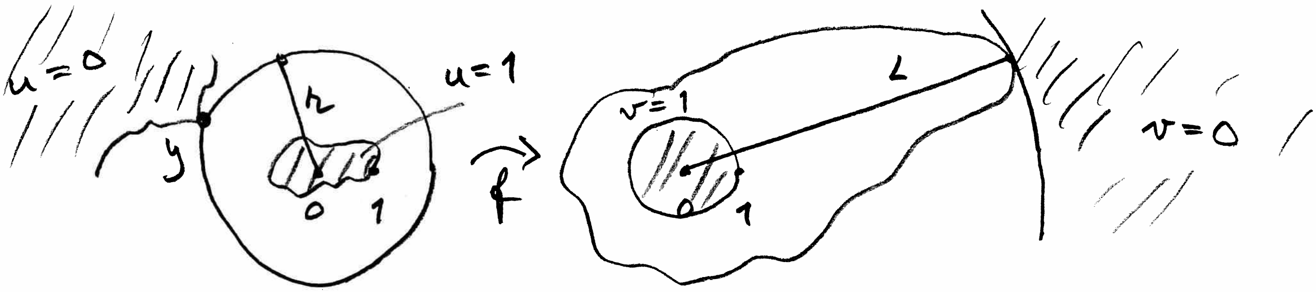
$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right) \qquad \forall x,y,z \in \mathbb{C}.$$

Composing with similarities (which do change conditions (3.9)) may assume $x = 0 = f(x)$, $z = 1 = f(z)$. Thus need:

$$|f(y)| \leq \eta_K(|y|)$$

1°) $|y| = r \geq 1$.

Wlog, $|f(y)| = \max_{|\xi|=r} |f(\xi)| =: L > 1$.



Let $v(z) = \begin{cases} 0, & |z| \geq L \\ 1, & |z| \leq 1 \\ \dfrac{\log(|z|/L)}{\log L}, & 1 \leq |z| \leq L \end{cases}$ and $u = v \circ f$

Then

$$\int_{\mathbb{C}} |\nabla u|^2 = \int_{\mathbb{C}} |\nabla (v \circ f)|^2 \underset{\text{Le 3.2}}{\leq} K \int_{\mathbb{C}} |\nabla v|^2 = K \int_1^L \frac{2\pi}{(\log L)^2} \frac{dt}{t}$$

$\Rightarrow$

$$(3.10) \qquad \int_{\mathbb{C}} |\nabla u|^2 \leq \frac{2\pi K}{\log L}$$

To get a lower bound for $\int_{\mathbb{C}} |\nabla u|^2$, let $E = f^{-1}(\overline{B(0,1)})$ and $F = f^{-1}(\mathbb{C} \setminus B(0,L))$. Then:

i) $u|_E \equiv 1$, $u|_F \equiv 0$

ii) E, F connected; $\{0,1\} \subset E$; $F \cap B(0,r) \neq \emptyset$. F unbounded

iii) $u \in W^{1,2}(\mathbb{C})$ & u continuous.

To use oscillation lemma for a lower bound need also:

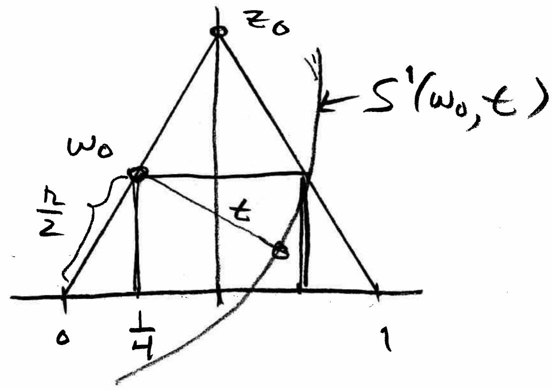
3.23. Lemma. Let $|z_0| = r \geq 1$. Then $\exists\, w_0 \in \mathbb{C}$ s.t. for $\frac{r}{2} < t < \sqrt{1/2 + r^2/4}$;

- $S^1(w_0, t) = \{\zeta : |\zeta - w_0| = t\}$ separates $0, 1$

and

- $S^1(w_0, t)$ separates z_0 from ∞.

Proof: Let first $|z_0| \leq |1 - z_0|$, take then $w_0 = \frac{z_0}{2}$.
A geometric consideration shows that extremal case is when $|z_0| = |1 - z_0|$



$$\Rightarrow \quad |w_0 - 1|^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{r}{2}\right)^2 - \frac{1}{4}$$

Thus in general

$$|w_0 - 1| \geq \sqrt{\left(\frac{r}{2}\right)^2 + \frac{1}{2}}$$

and for $\frac{r}{2} < t < \sqrt{\left(\frac{r}{2}\right)^2 + \frac{1}{2}}$, $S^1(w_0, t)$ separates $0, 1$ and z_0, ∞.

(If $|z_0 - 1| \leq |z_0|$, claim follows by symmetry.) $\square$

Return to proof of Thm. 3.22: By Oscillation Lemma,

$$\frac{(2\pi)^2 K}{\log L} \geq 2\pi \int_{\mathbb{C}} |\nabla u|^2 \geq \int_{r/2}^{\sqrt{(r/2)^2 + 1/2}} \underbrace{\operatorname{osc}_u (S^1(w_0, t))}_{\geq 1} \frac{dt}{t}$$

$$= \frac{1}{2} \log\left(1 + \frac{1}{2r^2}\right)$$

$$\Rightarrow \quad L \leq \exp\left(\frac{8\pi^2 K}{\log\left(1 + \frac{1}{2r^2}\right)}\right) = \eta_K(r), \quad r \geq 1.$$

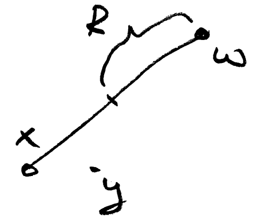
So we have shown: $\dfrac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta_K\left(\dfrac{|x-y|}{|x-z|}\right)$ when

$\dfrac{|x-y|}{|x-z|} \ge 1$, $x, y, z \in \mathbb{C}$.

In particular (since f homeo) f satisfies the condition

(3.11) $\quad |z_1 - w| \le |z_2 - w| \Rightarrow |f(z_1) - f(w)| \le H\,|f(z_2) - f(w)|$

where $H = \eta_K(1)$. This cond. is called weak quasi-symmetry. We show that for maps of $\mathbb{C}$, weak qs $\Rightarrow$ qsymmetry.



For this, if $|x-y| \le \frac{1}{2}|w-x| = R$,

Let $w_1 = \dfrac{w+x}{2}$ be midpoint of $[x,w]$.

Then weak quasisymmetry $\Rightarrow$

$$|f(x)-f(y)| \le H^2 |f(w)-f(w_1)| \le H^3 \min_{|\zeta - w| = R} |f(\zeta) - f(w)|$$

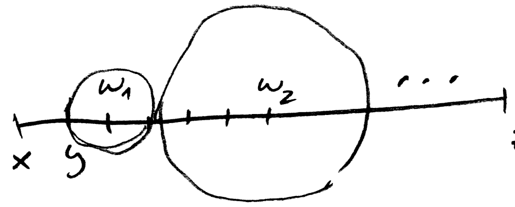
Thus $f\,B(w,R)$ contains disk of radius $\dfrac{|f(x)-f(y)|}{H^3}$

Next, if $|x-z| \equiv \rho = a^n |x-y|$, $3 < a < 9$,

can find disjoint disks

$B_j = B(w_j, R_j)$ s.t.

$R_j = \frac{1}{2}|w_j - x| > |x-y|$, $j = 1, \dots, n$



Since disks disjoint,

$$n\pi\,|f(x)-f(y)|^2 \le H^6 \sum_{j=1}^{n} |fB_j| \le H^6\,|f B(x,\rho)| \le$$

$$\leq \pi H^6 \max_{|\zeta - x| = \rho} |f(\zeta) - f(x)|^2 \leq \pi H^8 |f(z) - f(x)|^2$$

Thus

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \frac{H^4}{\sqrt{n}} = \frac{H^4 \log a}{\left(\log \frac{|x-y|}{|x-z|}\right)^{1/2}} \to 0 \quad \text{as } \frac{|x-y|}{|x-z|} \to 0$$

So can take $\eta(t) = \dfrac{H^4 \log a}{(\log t)^{1/2}}$ when $0 < t < \frac{1}{3}$,

and have produced $\eta = \eta_K(t)$ for $t \geq 1$ & $0 < t < 1/3$

- On the remaining interval $\frac{1}{3} < t < 1$ can interpolate as you like; in any case have shown that any K-qc $f: \mathbb{C} \to \mathbb{C}$ is η_K-quasisymmetric.

$\square$

IV Basic Properties of Quasiconformal Maps

- Starting from Definition (3.9.a) – (3.9.c) want to prove:

4.1. Theorem If $f: \Omega \to \Omega'$ K-quasiconf. and $g: \Omega' \to \Omega''$ is K'-quasiconformal, then

(i) $f^{-1}: \Omega' \to \Omega$ is K-quasiconformal

(ii) $g \circ f: \Omega \to \Omega''$ is $K \cdot K'$-quasiconformal

(iii) $J(z, f) > 0$ for a.e. $z \in \Omega$.

(iv) $|E| = 0 \iff |f(E)| = 0$ $\forall$ measurable $E \subset \Omega$.

Modulus of curve families

<u>def</u> Γ family of curves in $\Omega \subset \mathbb{R}^2$

$\rho : \Omega \to [0,\infty]$ Borel function admissible if

$$\int_\gamma \rho \, ds \geq 1 \quad \text{for every (rectifiable) } \gamma \in \Gamma$$

The modulus of Γ is

$$M(\Gamma) = \inf_{\rho \text{ adm.}} \int_\Omega \rho^2 \, dm$$

<u>Proposition</u> $f : \Omega \to \Omega'$ is ~~quasi~~conformal

for $\Gamma \subset \Omega$ $\quad f\Gamma = \{ f(\gamma) : \gamma \in \Gamma \} \subset \Omega'$

$$M(f\Gamma) = M(\Gamma)$$

<u>Proof.</u> $\tilde{\rho}$ admissible for $f\Gamma$

$$\rho(z) = \tilde{\rho}(fz)|f'(z)|$$

$$\int_\gamma \rho \, ds = \int_\gamma \tilde{\rho}(fz)|f'(z)|\,|dz| = \int_{f\gamma} \tilde{\rho}\, ds \geq 1$$

$$\Rightarrow \int_\Omega \rho^2 \, dm = \int_\Omega \tilde{\rho}^2(fz)|f'(z)|^2 \, dz = \int_{\Omega'} \tilde{\rho}^2 \, dm$$

take inf over all $\tilde{\rho}$ admissible

$$\Rightarrow \quad M(\Gamma) \leq M(f\Gamma)$$

Considering $f^{-1} : \Omega' \to \Omega \quad \Rightarrow \quad M(\Gamma) \geq M(f\Gamma)$ □

Prop. $f:\Omega \to \Omega'$ K-quasiconformal

$\frac{1}{K} M(\Gamma) \le M(f\Gamma) \le K\, M(\Gamma)$ for any curve family $\Gamma \subset \Omega$

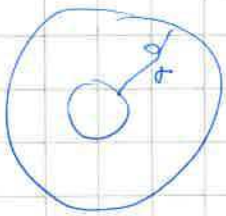
This is in fact characterizes K-quasiconformality

Modulus / Extremal length give an alternative approach to develop the theory.

This is implicit in our covering

$\rho \rightsquigarrow |\nabla u|$

Example $\Omega = \{1 < |z| < L\}$



$\Gamma = \{$curves joining boundary components$\}$

$M(\Gamma) = \dfrac{2\pi}{\log \frac{1}{L}}$ $\Leftarrow$ Exercise

length-area method

$\int_\gamma \rho\, ds \ge 1$ vs $\int_\Omega \rho^2\, dx$

Theorem 4.1 gives the basic analytic properties of quasiconformal maps. Main ingredient in the proof is of course Thm 3.22., but need also:

<u>4.2. Lemma</u> ("localization principle") If $f:\Omega\to\Omega'$ is K-quasiconformal and $\overline{B(z_0,2r)}\subset\Omega$, then

$$f|_{B(z_0,r)} \text{ is } \eta_K\text{-quasisymm.}$$

In fact, for $z\in B(z_0,r)$, $f(z)=\varphi\circ g(z)$ where

- $g:\mathbb{C}\to\mathbb{C}$ K-quasiconf, $g\,B(z_0,2r)=B(z_0,2r)$
- φ conformal on $B(z_0,2r)$

<u>Proof:</u> WLog, $B(z_0,2r)=\mathbb{D}=\{|z|<1\}$

Let $h: f\mathbb{D}\to\mathbb{D}$ be conformal with $h\circ f(0)=0$ $\left(\begin{smallmatrix}\text{Riemann}\\ \text{map}\end{smallmatrix}\right)$

Then $h\circ f$ is K-qconformal (Check !!)

and $h\circ f:\overline{\mathbb{D}}\to\overline{\mathbb{D}}$ homeo (Caratheodory)

Extend this to all of $\mathbb{C}$ by the reflection:

Set $\Phi(z)=1/\bar z$ and define

$$g(z)=\begin{cases} h\circ f(z), & |z|<1\\ \Phi\circ h\circ f\circ\Phi(z), & |z|>1\end{cases}$$

For $g\in W^{1,2}_{loc}$ see: Evans-Gariepy p.130

Then $g:\mathbb{C}\to\mathbb{C}$ is K-quasiconformal and by $g(\mathbb{D})=\mathbb{D}$, $g(0)=0$, $g(\infty)=\infty$

c) $f|_{\mathbb{D}} = \varphi \circ g|_{\mathbb{D}}$, $\varphi = h^{-1} : \mathbb{D} \to f\mathbb{D}$ conformal

Poincare distance

d) $\varrho_{\mathbb{D}}(g(z), 0) \le M_1 = M_1(K)$, when $|z| < \frac{1}{2}$

[Proof of d): If $|z| < \frac{1}{2}$, $|\xi| = 1 \Rightarrow \frac{|g(0) - g(\xi)|}{|g(z) - g(\xi)|} \le \eta_K(2)$

$\Rightarrow \operatorname{dist}(g(z), S^1) = \delta \ge 1/\eta_K(2) > 0 \Rightarrow$

$$\varrho_{\mathbb{D}}(g(z), 0) = \log\left(\frac{1 + |g(z)|}{1 - |g(z)|}\right) \le \log \frac{2}{\delta}$$]

Now, for $z, w, x \in B(0, 1/2)$

Thm. 2.7

$$\frac{|f(z) - f(w)|}{|f(x) - f(w)|} = \frac{|\varphi \circ g(z) - \varphi \circ g(w)|}{|\varphi \circ g(x) - \varphi \circ g(w)|} \le$$

$$e^{M(K)} \frac{|g(z) - g(w)|}{|g(x) - g(w)|} \le e^{M(K)} \eta_K\left(\frac{|z - w|}{|x - w|}\right)$$

Thm 3.22. $\square$

<u>Proof of Thm. 4.1</u>: By localization principle,

$f|_{B(z_0, r/2)}$ η_K-quasisymmetric, when $B(z_0, 2r) \subset \Omega$;

thus $|E| = 0 \Leftrightarrow |f(E)| = 0$ By Thm. 3.17.

If $E = \{z \in \Omega : J(z, f) = 0\}$, By Corollary 3.18,

$|fE| = \int_E J(z, f) = 0 \overset{\text{Thm 3.17}}{\Longrightarrow} |E| = 0$. Thus iii) & iv) hold.

i) $f|_{B(z_0,r)}$ q-symm. $\Rightarrow$ $f^{-1}|_{fB(z_0,r)}$ q-symm.

$\underset{\text{Thm 3.14}}{\Rightarrow}$ f^{-1} $\tilde{K}$-qc, for some $\tilde{K} < \infty$.

But if $z = f(w)$ and f differentiable at w, f^{-1} at z, then

$$Id = Df(w)\, Df^{-1}(z) \Rightarrow$$

$$|Df^{-1}(z)|^2 = |(Df(w))^{-1}|^2 = \frac{|Df(w)|^2}{J(w,f)^2}$$

$$\leq \frac{K}{J(w,f)} = K\, J(z, f^{-1}) \Rightarrow f^{-1} \text{ is } K\text{-qc}.$$

ii) As in i) $g \circ f$ locally q-symm. $\Rightarrow$ qconf.

For a.e. z,

$$|D(g\circ f)(z)|^2 \underset{\text{Cor. 3.18}}{\leq} |Dg(fz)|^2\, |Df(z)|^2$$

$$\leq K'\, J(fz, g)\, K\, J(z,f) = K'K\, J(z, g\circ f).$$

$\square$

Hölder-Continuity

As the last "basic" property let us prove that K-quasiconformal maps are $1/K$-Hölder continuous. We give a proof using the isoperimetric inequality:

4.3. Theorem (Isoperimetric Inequality). Suppose $\Omega \subset \mathbb{C}$ is a Jordan domain with $\partial\Omega$ rectifiable. Then

$$|\Omega| \leq \frac{1}{4\pi} \mathcal{H}^1(\partial\Omega)^2$$

Here $\mathcal{H}^1$ = 1-dim. Hausdorff measure ≡ length. For a proof using Fourier series see e.g. [A-I-M], p. 80.

Next, let us denote

$$\lambda(K) := \sup\{|f(e^{i\theta})| : f \text{ K-qconf on } \mathbb{C},\ f(0)=0,\ f(1)=1\}$$

$$(= \eta_K(1) \text{ for "best" } \eta_K)$$

It is known, that $1 \leq \lambda(K) \leq \frac{1}{16} e^{\pi K}$ and $\lambda(K) \to 1$ as $K \to 1$.

4.4. Theorem. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal with $f(0)=0$, $f(1)=1$, then

$$|f(z)| \leq \lambda(K)^2 |z|^{1/K}, \qquad |z| < 1.$$

Moreover, $f(z) = \frac{z}{|z|}|z|^{1/K}$ is K-quasiconformal (see the next section), so that the Hölder exponent $\frac{1}{K}$ optimal.

Proof: Let $B = B(0,t)$ and write $J = J(z,f)$. Then by Isoperimetric inequality, for a.e. $t \in (0,1)$ (i.e. when $f|_{\partial B}$ is abs. cont.)

$$\int_B J \leq \frac{1}{4\pi}\Big(\int_{\partial B} |Df|\Big)^2 \underset{\text{Hölder}}{\leq} \frac{2\pi t}{4\pi}\int_{\partial B} |Df|^2 \leq \frac{Kt}{2}\int_{\partial B} J.$$

Prop. $f: \Omega \to \Omega'$ K-quasiconformal

$$\frac{1}{K} M(\Gamma) \le M(f\Gamma) \le K\, M(\Gamma) \qquad \text{for any curve family } \Gamma \subset \Omega$$

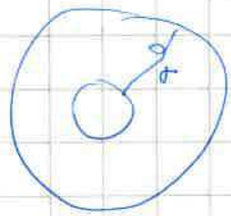
This in fact characterizes K-quasiconformality

Modulus / Extremal length give an alternative approach to develop the theory.

This is implicit in our covering

$$\rho \rightsquigarrow |\nabla u|$$

Example $\Omega = \{ 1 < |z| < L \}$



$\Gamma = \{$ curves joining boundary components $\}$

$$M(\Gamma) = \frac{2\pi}{\log \frac{1}{L}} \qquad \Leftarrow \quad \text{Exercise}$$

length-area method

$$\int_\gamma \rho \, ds \ge 1 \quad \text{vs} \quad \int_\Omega \rho^2 \, da$$

We have thus shown that $\phi(t) := \int_{B(0,t)} J(z,f)$ satisfies

$$\phi(t) \le \frac{Kt}{2}\phi'(t), \quad \text{(a.e.)}\ 0 < t \le 1.$$

$$\Rightarrow \frac{d}{dt}\left(t^{-2/K}\phi\right) = t^{-2/K-1}\left(t\phi' - \frac{2}{K}\phi\right) \ge 0, \text{ and as } \phi$$

is abs. cont. $\Rightarrow$

$$\phi(t) \le t^{2/K}\phi(1), \quad 0 < t \le 1.$$

Now, if $|z| = r < 1$,

$$|f(z)| \le \lambda(K) \inf_{|\zeta|=r} |f(\zeta)| \le \lambda(K)\left(\frac{1}{\pi}|fB(0,r)|\right)^{1/2}$$

$$= \lambda(K)\left(\frac{1}{\pi}\phi(r)\right)^{1/2} \le \lambda(K)\, r^{1/K}\left(\frac{1}{\pi}|fB(0,1)|\right)^{1/2}$$

$$\le \lambda(K)\, r^{1/K}\left(\max_{|\zeta|=1}|f(\zeta)|^2\right)^{1/2} \le \lambda(K)^2\, r^{1/K}. \quad \square$$

4.5. Corollary. If $f: \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, then f is η-quasisymmetric, where

$$\eta(t) = C(K)\max\{t^K, t^{1/K}\}.$$

Proof: Exercises 2. $\square$

4.6. Corollary. If $f: \Omega \to \Omega'$ is K-quasiconformal, then f is locally $1/K$-Hölder continuous. In fact, if $2B \subset \Omega$ then

$$|f(z) - f(w)| \le C(K)\operatorname{diam}(fB)\frac{|z-w|^{1/K}}{\operatorname{diam}(B)^{1/K}}; \quad z, w \in B.$$

Proof: Follows from Corollary 4.5., localization principle = Lemma 4.2 and from Theorem 2.7:

$$f(z) = \underbrace{\varphi}_{\text{conf. in } 2B} \circ \underbrace{g(z)}_{K\text{-qc in } \mathbb{C}} \quad , z \in B \quad . \qquad \square$$

V PDE's and Quasiconformal maps in the Complex Setting.

V.1. Quasiregular Maps and Complex Notation

5.1. Definition. If $f \in W^{1,2}_{loc}(\Omega)$ satisfies

(5.1.a) $\quad \max_\alpha |\partial_\alpha f(z)| \leq K \min_\alpha |\partial_\alpha f(z)| \quad$ a.e. $z \in \Omega$

and

(5.1.b) $\quad J(z,f) \geq 0 \quad$ a.e. $z \in \Omega$,

then we call f K-quasiregular.

Remarks 1°) f K qconformal $\Leftrightarrow$ f K-qregular homeo.

2°) We will later show that each K-quasiregular map is continuous and open $\underset{\text{Gehring-Lehto}}{\Rightarrow}$ Differentiable a.e.

3°) Even if differentiability a.e. is not given by the definition, we can still define the notions in (5.1): Set $f = u+iv \Rightarrow J(z,f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ and $\partial_\alpha f(z) = \cos(\alpha)\,\partial_x f(z) + \sin(\alpha)\,\partial_y f(z)$.

In the sequel it will convenient to use complex derivatives $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$.

E.g., by Exercise 1, if f differentiable at z, then

$$Df(z)h = \underbrace{\partial f(z)h}_{\substack{\mathbb{C}-\text{linear}\\ \text{in } h}} + \underbrace{\bar\partial f(z)h}_{\substack{\text{anti-linear}\\ \text{in } h}}, \quad J(z,f) = |\partial f|^2 - |\bar\partial f|^2$$

Thus

$$\max_\alpha |\partial_\alpha f(z)| = |\partial f(z)| + |\bar\partial f(z)|$$

$$\min_\alpha |\partial_\alpha f(z)| = |\partial f(z)| - |\bar\partial f(z)| \qquad (\text{when } J(z,f) \geq 0)$$

so that (5.1.a) – (5.1.b) $\Longleftrightarrow$

$$|\partial f(z)| + |\bar\partial f(z)| \leq K\big(|\partial f(z)| - |\bar\partial f(z)|\big) \Longleftrightarrow |\bar\partial f| \leq \frac{K-1}{K+1}|\partial f|.$$

But this is <u>equivalent</u> to

(5.2)
$$\begin{cases} \bar\partial f(z) = \mu(z)\,\partial f(z) & \text{for a.e. } z \in \Omega, \text{ where} \\ |\mu(z)| \leq k := \dfrac{K-1}{K+1} < 1 & \text{—''—} \end{cases}$$

<u>Remarks:</u>

- (5.2) is called the <u>Beltrami equation</u> (and has central role in the sequel)
- (5.2) is a <u>$\mathbb{C}$ – linear</u> equation for $f = u + iv$, and it is a linear <u>system</u> of <u>PDE's</u> for u, v.
- A K-quasiregular map $f \equiv W^{1,2}_{loc}$-solution to $\bar\partial f = \mu\,\partial f$; $|\mu| \leq \frac{K-1}{K+1}$

5.2 <u>Example.</u> Let $f(z) = z|z|^{1/K-1} = \frac{z}{|z|}|z|^{1/K}$.

Then $\partial f(z) = \frac{1}{2}\left(\frac{1}{K}+1\right)|z|^{\frac{1}{K}-1}$ and $\bar\partial f(z) = \frac{1}{2}\left(\frac{1}{K}-1\right)\frac{z}{\bar z}|z|^{\frac{1}{K}-1}$,

(Check!)

and we have

$$\bar\partial f(z) = \frac{1-K}{1+K}\,\frac{z}{\bar z}\,\partial f(z).$$

Also $f \in W^{1,2}_{loc}(\mathbb{C})$ & homeo

Thus f is K-quasiconformal.

<u>Remark</u> For solutions to $\bar\partial f = \mu\,\partial f$ we call the coefficient $\mu(z) = \mu_f(z)$ the <u>complex</u> <u>dilatation</u> of f. Note that for quasiconformal maps (at least) $J_f(z) > 0$ a.e $\Rightarrow |\partial f|^2 - |\bar\partial f|^2 > 0$ a.e $\Rightarrow \partial f(z) \neq 0$ for a.e. z. Thus $\mu(z) := \bar\partial f(z)/\partial f(z)$ uniquely defined almost

d) Another basic case is when $\sigma(x) = \sigma(x)^t$ and $\det \sigma \equiv 1$.
This is exactly the case where $\nu(z) \equiv 0$, i.e. $f = u + iv$ satisfies the ($\mathbb{C}$-linear) equation $\bar\partial f = \mu\, \partial f$; c.f. Exercises 2.

VI. Measurable Riemann Mapping Theorem.

The goal of this is to show:

If $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \le k < 1$, then there is a homeo $f : \mathbb{C} \to \mathbb{C}$ with

- $f \in W^{1,2}_{loc}(\mathbb{C})$
- $\bar\partial f = \mu(z)\, \partial f$ a.e. $z \in \mathbb{C}$
- f is unique up to postcomposition with a similarity.

VI.1. Uniqueness

6.1. Weyl's Lemma. Suppose $f \in W^{1,1}_{loc}(\Omega)$. TFAE

i) f is analytic (i.e. f has analytic representative)

ii) $\bar\partial f = 0$ in the weak sense, i.e. $\int_\Omega \bar\partial \varphi \cdot f \, dm = 0 \;\; \forall \varphi \in C_0^\infty(\Omega)$

iii) $\bar\partial f(z) = 0$ for a.e. $z \in \Omega$.

Remarks: Without condition $f \in W^{1,1}_{loc}(\Omega)$, $\bar\partial f = 0$ a.e $\not\Rightarrow$ f analytic;

simply take $f(z) = \frac{1}{z}$. BUT if we assume only $f \in L^1_{loc}(\mathbb{C})$ AND that $\bar{\partial} f = 0$ in the weak sense, this gives f analytic ! (see proof below)

Proof of Weyl's Lemma: Clearly i) $\Rightarrow$ ii), iii). Also, by Theorem 3.3, ii) $\Leftrightarrow$ iii). Thus suffices to prove ii) $\Rightarrow$ i).

For this, take $0 \le \varphi \in C_0^\infty(\mathbb{C})$ with $\int_\Omega \varphi(z)\, dm = 1$. Assume, say, that $\operatorname{supp}(\varphi) \subset \mathbb{D}$, and denote $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^2} \varphi\left(\frac{z}{\varepsilon}\right)$.

From Real Analysis, we know: $\varphi_\varepsilon * f \in C^\infty$ and $\varphi_\varepsilon * f \to f$ (in $L^1_{loc}(\Omega)$)

More precisely, let $\Omega_\varepsilon = \{ z \in \Omega : d(z, \partial\Omega) > \varepsilon \}$.

Then for $z \in \Omega_\varepsilon$,

$$\bar{\partial}(\varphi_\varepsilon * f)(z) = \bar{\partial} \int_\Omega \varphi_\varepsilon(z-y) f(y)\, dm = \int_\Omega \bar{\partial} \varphi_\varepsilon(z-y) f(y) \overset{\text{by (ii)}}{=} 0$$

Hence $\varphi_\varepsilon * f$ analytic in Ω_ε. Further, if $z \in \Omega_{2\varepsilon}$, let $B(z) = B(z, d(z, \partial\Omega)/2)$. By mean value property of analytic functions, $(\varphi_\varepsilon * f)(z) = \fint_{B(z)} (\varphi_\varepsilon * f)\, dm$, $z \in \Omega_{2\varepsilon}$

Thus
$$\left| (\varphi_\varepsilon * f)(z) - (\varphi_{\tilde{\varepsilon}} * f)(z) \right| \le \frac{1}{|B(z)|} \int_{B(z)} |\varphi_\varepsilon * f - \varphi_{\tilde{\varepsilon}} * f| \to 0 \quad (\varepsilon, \tilde{\varepsilon} \to 0)$$

as $\varphi_\varepsilon * f \to f$ in $L^1(B(z))$.

Convergence is uniform on compact subsets of Ω,

$\Rightarrow g := \lim_{\varepsilon \to 0} (\varphi_\varepsilon * f)(z)$ analytic in Ω, and

$$\fint_{B(z)} |g(z) - f(z)| = 0 \quad \forall z \Rightarrow f = g \quad \text{a.e. } z \in \Omega. \ \square$$

6.2. Remark. In particular, from Weyl's lemma and (5.2),

f 1-quasiconformal $\iff$ f conformal

Next we need to analyse the complex dilatation of a composition of quasiconformal maps $g\circ f$. By chain rule (Exercise)

$$\partial(g\circ f) = \partial g(f)\,\partial f + \bar\partial g(f)\,\overline{\bar\partial f} \ ; \quad \bar\partial(g\circ f) = \partial g(f)\,\bar\partial f + \bar\partial g(f)\,\overline{\partial f}$$

One can write this in a matrix form:

$$\begin{pmatrix} \bar\partial(g\circ f) \\ \partial(g\circ f) \end{pmatrix} = \begin{pmatrix} \overline{\partial f} & \bar\partial f \\ \overline{\bar\partial f} & \partial f \end{pmatrix} \begin{pmatrix} \bar\partial g(f) \\ \partial g(f) \end{pmatrix}$$

and inverting the matrix gives

$$\bar\partial g(f) = \frac{1}{J(z,f)}\left(\partial f \cdot \bar\partial(g\circ f) - \bar\partial f \cdot \partial(g\circ f)\right)$$

$$\partial g(f) = \frac{1}{J(z,f)}\left(\overline{\partial f}\ \partial(g\circ f) - \overline{\bar\partial f} \cdot \bar\partial(g\circ f)\right)$$

Dividing (recall $J(z,f)>0$ for a qc map) we have

6.3. Corollary. If $f:\Omega\to\Omega'$ and $g:\Omega'\to\Omega''$ are quasiconformal, then

$$\mu_g(f) = \frac{\mu_{g\circ f} - \mu_f}{1-\mu_{g\circ f}\,\overline{\mu_f}}\ \frac{\partial f}{\overline{\partial f}} \quad \text{for a.e. } z\in\Omega.$$

In other words, if we write

$$T_a(z) = \frac{z-a}{1-\bar a z}, \quad T_a : \mathbb{D}\to\mathbb{D},\ |a|<1,$$

then Corollary 6.3 gets the form:

$$\mu_g(f) = T_{\mu_f}(\mu_{g\circ f}) \frac{\partial f}{\partial f}$$

<u>6.4. Corollary</u> (Uniqueness part of M.R.M.T) Suppose f and h are quasiconformal in a domain $\Omega \subset \mathbb{C}$. Then following are equivalent:

(i) $\mu_f(z) = \mu_h(z)$ for a.e. $z \in \Omega$

(ii) $h = g \circ f$, where g conformal in $f(\Omega)$.

<u>Proof:</u> ii) $\Rightarrow$ i): g smooth $\Rightarrow$ chain rule applies,

$\bar\partial h = g'(f)\,\bar\partial f$, $\partial h = g'(f)\,\partial f$ $\Rightarrow$ i).

i) $\Rightarrow$ ii): Since f homeo, can define $g := h \circ f^{-1}$.

Theorem 4.1 $\Rightarrow$ g qconf. in $f(\Omega)$, and Corollary 6.3 $\Rightarrow$

$$\mu_g(fz) = \frac{\mu_h - \mu_f}{1 - \mu_h \overline{\mu_f}} \frac{\partial f}{\partial f} = 0 \quad \text{a.e} \Rightarrow \mu_g = 0 \quad \text{a.e.}$$

and by Remark 6.2 $\Rightarrow$ g conformal. $\square$

<u>6.5. Remark</u> If $\overbrace{f, h}^{W^{1,2}\text{ homeos}} : \mathbb{C} \to \mathbb{C}$ satisfy $\bar\partial f = \mu \partial f$, $\bar\partial h = \mu \partial h$, i.e. same Beltrami eqn, then Cor. 6.4 $\Rightarrow$

$$h = \phi \circ f, \quad \text{where } \phi(z) = az + b \text{ a similarity!}$$

VI.2. Tools for Proving Existence in M.R.M.T

If $f \in C_0^1(\mathbb{C})$, then

$$(6.1) \qquad f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar\partial f(\zeta)}{(\zeta - z)}\, dm(\zeta) \quad , \quad z \in \mathbb{C}.$$

Indeed, recall Green's formula, i.e. Lemma 2.5:

$\frac{1}{2i}\int_{\partial\Omega} F(\zeta)\, d\zeta = \int_{\Omega} \bar\partial F\, dm$, $F \in C^1(\bar\Omega)$. If we let here $\Omega = B(z,R) \setminus B(z,\varepsilon)$ and $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ and R so large that $B(z,R)$ contains $\mathrm{supp}(f)$, then: $-\frac{1}{\pi}\frac{1}{2i}\int_{\partial\Omega} F(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \to f(z)$ when $\varepsilon \to 0$ $\left[\text{as } \frac{1}{2\pi i}\int_{|\zeta - z|=\varepsilon} \frac{d\zeta}{\zeta - z} = 1\right]$.

The identity gives rise to following notion

6.6. Definition. The Cauchy transform

$$(6.2) \qquad \mathcal{C}(h)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(\zeta)}{\zeta - z}\, dm(\zeta) \quad , \quad h \in C_0(\mathbb{C}), \; z \in \mathbb{C}.$$

The integral (6.2) converges, since $\zeta \mapsto \frac{1}{\zeta - z} \in L^q_{loc}(\mathbb{C})\ \forall q < 2$.

And by (6.1), $f = \mathcal{C}(\bar\partial f)$ whenever $f \in C^1(\mathbb{C})$ has compact support.

6.6. Remarks. 1°) $\mathcal{C} : C_0^\infty(\mathbb{C}) \to C^\infty(\mathbb{C})$

Indeed, since $\mathcal{C}$ is a convolution operator, it commutes with all derivatives, $\partial_{x_j} \mathcal{C} = \mathcal{C} \partial_{x_j}$ [check this!].
But $\mathcal{C}(h)$ need not have compact support even if h does. (e.g. if $\int_{\mathbb{C}} h \, dm \neq 0$). On the other hand, $\mathcal{C}(h)(z)$ is analytic outside the support of h.

2°) For $g \in C_0^1(\mathbb{C})$,

$$(6.3) \qquad \bar\partial[\mathcal{C}(g)] = \mathcal{C}(\bar\partial g) \overset{(6.1)}{=} g .$$

In other words, $\boxed{\bar\partial \circ \mathcal{C} = \mathcal{C} \circ \bar\partial = Id}$ (on $C_0^1(\mathbb{C})$ at least)

6.7. Definition. For $g \in C_0^\infty(\mathbb{C})$ the Beurling transform is defined by

$$(6.4) \qquad S(g) := \partial[\mathcal{C}(g)] .$$

Remark For $g \in C_0^\infty(\mathbb{C})$, $z \in \mathbb{C}$, we have

$$(Sg)(z) = \partial(\mathcal{C}g)(z) = \mathcal{C}(\partial g)(z) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\mathbb{C}\setminus B(z,\varepsilon)} \frac{\partial g(\zeta)}{\zeta - z} \, dm$$

$$\overset{\text{int. by parts}}{=} \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{\mathbb{C}\setminus B(z,\varepsilon)} \frac{g(\zeta)}{(\zeta - z)^2} \, dm + \lim_{\varepsilon \to 0} -\frac{1}{\pi} \int_{|\zeta - z| = \varepsilon} \frac{g(\zeta)}{\zeta - z} \, d\bar\zeta,$$ where

we have used the Green's formula related to ∂-derivative,

$$\int_{\Omega} \partial F(\zeta)\, dm = -\frac{1}{2i} \int_{\partial\Omega} F(\zeta)\, d\bar{\zeta}$$

(follows from Lemma 2.5 by taking the complex conjugate)

Above

$$\int_{|\zeta - z| = \varepsilon} \frac{g(\zeta)}{\zeta - z} d\bar{\zeta} = \underbrace{\int_{|\zeta - z| = \varepsilon} \frac{g(\zeta) - g(z)}{\zeta - z} d\bar{\zeta}}_{\leq \|g\|_{Lip_1} |2\pi\varepsilon| \to 0} + \underbrace{g(z) \int_{|\zeta - z| = \varepsilon} \frac{d\bar{\zeta}}{\zeta - z}}_{g(z) \int_0^{2\pi} \frac{-i e^{-i\theta} d\theta}{\varepsilon e^{i\theta}} = 0\,!} \xrightarrow{\text{as } \varepsilon \to 0} 0$$

Thus for $g \in C_0^\infty(\mathbb{C})$

$$(6.5) \quad (Sg)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon} \frac{g(\zeta)}{(\zeta - z)^2} dm =: -\frac{1}{\pi}\, \text{p.v.} \int_{\mathbb{C}} \frac{g(\zeta)}{(\zeta - z)^2} dm$$

where p.v. stands for "principal value", defined by (6.5).
Indeed the kernel $\frac{1}{(\zeta - z)^2}$ is too singular (i.e. $\notin L^1_{loc}$) for the integral $\int_{\mathbb{C}} g(\zeta)/(\zeta - z)^2$ to exist in the usual sense.
But the principal value converges at every z when $g \in C_0^\infty(\mathbb{C})$.
This can also be seen directly: If $g \in C_0^\infty(\mathbb{C})$ & $\text{supp}(g) \subset B(z, R)$,

$$\int_{\varepsilon < |\zeta - z|} \frac{g(\zeta)}{(\zeta - z)^2} = \int_{\varepsilon < |\zeta - z| < R} \frac{g(\zeta) - g(z)}{(\zeta - z)^2}$$

which converges when $\varepsilon \to 0$, since $\frac{g(\zeta) - g(z)}{(\zeta - z)^2} \in L^1_{loc}(\mathbb{C})$

why!?

In the theory of singular integrals the pointwise behaviour is usually controlled by maximal transforms such as

$$S_* f(z) = \frac{1}{\pi} \sup_{\varepsilon>0} \left| \int_{|\xi-z|>\varepsilon} \frac{f(\xi)}{(\xi-z)^2} \right|$$

absolutely convergent for any $\varepsilon>0$ & $f \in L^p(\mathbb{C})$, $1<p<\infty$ (Hölder)

An alternative and elementary argument for the Beurling transf due to Mateu and Verdera:

Observe that S is symmetric,

$$\int_{\mathbb{C}} \eta(z)(S\phi)(z) = -\frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{|z-\xi|>\varepsilon} \frac{\eta(z)\phi(\xi)}{(\xi-z)^2} = \int_{\mathbb{C}} \phi(\xi)(S\eta)(\xi)$$

for $\eta, \phi \in C_0^\infty(\mathbb{C})$

By continuity it extends to all $\phi, \eta \in L^2(\mathbb{C})$

We will see later that

$$\cancel{S(\chi_{B(0,1)})} \quad S\chi_{\mathbb{D}}(z) = \begin{cases} 0 & |z| \le 1 \\ -\frac{1}{z^2} & |z| > 1 \end{cases}$$

and

$$S(\chi_{B(a,r)})(z) = \begin{cases} 0 & |z-a| \le r \\ -\frac{r^2}{(z-a)^2} & |z-a| > r \end{cases}$$

Now using this & symmetry

$$-\frac{1}{\pi} \int_{|z-\xi|>\varepsilon} \frac{f(\xi)}{(\xi-z)^2} = \frac{1}{\pi\varepsilon^2} \int_{\mathbb{C}} S(\chi_{B(z,\varepsilon)}) f =$$

$$= \frac{1}{\pi\varepsilon^2}\int_{\mathbb{C}} \chi_{B(z,\varepsilon)}\, Sf = \frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)} Sf$$

$\longrightarrow Sf(z)$ for a.e. z by Lebesgue's differentiation theorem

So the principal value integral converges a.e. for $f \in L^2(\mathbb{C})$

Once we know boundedness on $L^p(\mathbb{C})$ the same argument works for L^p as well.

<u>Theorem</u> 4.0.10 (AIM) Suppose $f \in L^p(\mathbb{C})$ $1 < p < \infty$

The the limit $(Sf)(z) = -\frac{1}{\pi}\lim_{\varepsilon \to 0}\int_{|z-\zeta|\geq\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}$

exists at a.e. $z \in \mathbb{C}$. (~~at the Lebes~~

<u>Riesz - Thorin convexity theorem</u> $1 \leq p_1 \leq p_2 < \infty$ exponents

Suppose $T : L^{p_1}(\Omega,\mu) \cap L^{p_2}(\Omega,\mu) \to L^{p_1}(\Omega,\mu) \cap L^{p_2}(\Omega,\mu)$

is a linear operator such that

$$\|T\phi\|_{p_1} \leq \|T\|_{p_1}\|\phi\|_{p_1}$$
$$\|T\phi\|_{p_2} \leq \|T\|_{p_2}\|\phi\|_{p_2}$$

for every $\phi \in L^{p_1}(\Omega,\mu) \cap L^{p_2}(\Omega,\mu)$

Then T extends to a bounded linear operator for $p_1 \leq p \leq p_2$

$T : L^p(\Omega,\mu) \to L^p(\Omega,\mu)$ $\qquad \|Tf\|_p \leq \|T\|_{p_1}^{1-\theta}\|T\|_{p_2}^{\theta}\|f\|_p$

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

<u>Corollary</u> $p \mapsto \|T\|_p$ is locally Lipschitz continuous on (p_1, p_2)

6.8. Remarks a) It is possible to show (see [AIM, p.97]) that for $f \in L^p(\mathbb{C})$ the limit defining the principal value integral,

$$\text{p.v.} \int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2}\, dm := \lim_{\varepsilon \to 0} \int_{|\zeta - z| > \varepsilon} \frac{f(\zeta)}{(\zeta - z)^2} \quad \text{exists for a.e. } z \in \mathbb{C}$$

This makes Sf a well defined object for L^p-fcn's, too.

b) For $g \in C_0^\infty(\mathbb{C})$ we have

$$(6.6.) \qquad S(g_{\bar z}) = g_z \quad , \text{i.e. } "S \circ \bar\partial = \partial"$$

Indeed, $S(g_{\bar z}) := \partial \underbrace{\mathcal{C}(g_{\bar z})}_{= g \text{ by } (6.1)} = g_z$.

To extend (6.6) to Sobolev functions we need

6.9. Theorem $\|Sf\|_{L^2(\mathbb{C})} = \|f\|_{L^2(\mathbb{C})} \quad \forall f \in L^2(\mathbb{C})$.

Proof: If $g \in C_0^\infty(\mathbb{C})$, then

$$\int_{\mathbb{C}} |g_z|^2 = \int_{\mathbb{C}} g_z \bar g_{\bar z} \underset{\substack{\uparrow \\ \text{int. by parts}}}{=} \int_{\mathbb{C}} g_{\bar z} \bar g_z = \int_{\mathbb{C}} |g_{\bar z}|^2 .$$

(Thus have shown: $\int_{\mathbb{C}} J(z,g) = 0$ for $g \in C_0^\infty(\mathbb{C})$!)

So for $f = g_{\bar z}$, $g \in C_0^\infty(\mathbb{C})$, we have the claim. But such f's are dense in L^2: If $\underbrace{\ }_{(f \in L^2(\mathbb{C}) \text{ and}}$ $f \perp \{g_{\bar z} : g \in C_0^\infty(\mathbb{C})\}$ then

$$0 = \langle \varphi_{\bar z}, f \rangle_{L^2} = \int_{\mathbb{C}} \varphi_{\bar z} \, f \, dm \quad \forall \varphi \in C_0^\infty(\mathbb{C}), \text{ i.e.}$$

$\bar\partial f = 0$ in the weak sense. By the remark (p. 50) after Weyl's lemma it follows f analytic in $\mathbb{C}$. But as $f \in L^2(\mathbb{C}) \Rightarrow$ f bounded $\Rightarrow$ $f \equiv \text{const} = 0$.

By the density of $\{ g_{\bar z} : g \in C_0^\infty(\mathbb{C}) \}$ in $L^2(\mathbb{C})$, Beurling op. extends to an isometry of all of $L^2(\mathbb{C})$,

$$\int_{\mathbb{C}} |Sf|^2 \, dm = \int_{\mathbb{C}} |f|^2 \, dm \qquad \forall f \in L^2(\mathbb{C})$$

□

Note: One can use the isometry property to define Sf for all $f \in L^2$, by the density as above.

Much deeper lie the L^p-properties of S. We have

6.10. Theorem The Beurling transform extends to a continuous operator on $L^p(\mathbb{C})$, $1 < p < \infty$,

$$\| Sf \|_{L^p(\mathbb{C})} \le \|S\|_p \cdot \|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C}),$$

for some constant $\|S\|_p$ depending only on p, with $p \mapsto \|S\|_p$ continuous.

(for proof see e.g. [AIM])

Remark: The famous Iwaniec conjecture asserts that (take can)

$$\|S\|_p = \max\left\{ p-1, \frac{1}{p-1} \right\}, \quad 1 < p < \infty.$$

Remark. The L^p-properties of the Cauchy transform are little more "subtle". For instance, by theory of Riesz potentials

(6.7) $$\| \mathcal{C} f \|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \leq c_p \| f \|_{L^p(\mathbb{C})} , \quad \underline{1 < p < 2}$$

For $f \in L^p(\mathbb{C})$, $2 \leq p < \infty$, the integral $\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} dm(\zeta)$ may not converge at ∞ (as $\frac{1}{\zeta} \notin L^q(|\zeta| > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$). Thus we set now

$$\mathcal{C} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} f(\zeta) \underbrace{\left[\frac{1}{\zeta - z} - \frac{1}{\zeta} \chi_{\mathbb{C} \setminus \mathbb{D}}(\zeta) \right]}_{\overset{|\zeta|>1}{=} \frac{z}{\zeta(\zeta - z)} \in L^q(|\zeta| > 1)} dm(\zeta)$$

This differs from "old" $\mathcal{C} f(z)$ only by a constant; thus still $\bar{\partial}\, \mathcal{C}(f) = f$!

6.11. Corollary. If $f \in L^p(\mathbb{C})$ with $f(z) = 0$ for $|z| > R$, then

$$\| \mathcal{C} f \|_{L^p(B(0,2R))} \leq C(p) R \| f \|_p , \quad R > 1, \ 1 < p < \infty .$$

In particular, if $L^p_0(\mathbb{C}) = \{ f \in L^p(\mathbb{C}), \operatorname{supp}(f) \text{ bounded} \}$, then

$$\mathcal{C} : L^p_0(\mathbb{C}) \longrightarrow \mathring{W}^{1,p}(\mathbb{C}) = \{ f \in W^{1,p}_{loc}(\mathbb{C}) : \nabla f \in L^p(\mathbb{C}) \}$$

Proof: If $K_R(\zeta) = \frac{-1}{\pi} \frac{1}{\zeta} \chi_{B(0,3R)}(\zeta)$, then for $|z| < 2R$,

$$K_R * f(\zeta) = -\frac{1}{\pi}\int_{|\zeta - z|<3R} \frac{1}{\zeta - z} f(\zeta) = \mathcal{C}f(z) \quad . \text{ Thus}$$

$$\| \mathcal{C}f \|_{L^p(B(0,2R))} \le \| K_R * f \|_{L^p(\mathbb{C})} \le \underbrace{\| K_R \|_{L^1}}_{=CR} \| f \|_{L^p(\mathbb{C})} .$$

Further, as $\| \bar\partial \mathcal{C} f \|_{L^p} = \| f \|_{L^p}$ and $\| \partial \mathcal{C} f \|_{L^p} = \| Sf \|_{L^p} \le c \| f \|_{L^p}$ we have $\| \nabla \mathcal{C} f \|_{L^p} \le c \| f \|_{L^p} < \infty$. $\square$

<u>6.12. Corollary</u> $\quad S(g_{\bar z}) = g_z \quad \forall\, g \in \mathring{W}^{1,p}(\mathbb{C})$.

<u>Pf.</u> Follows from (6.6) & Theorem 6.10 . $\square$

Corollary 6.12 makes it possible to calculate $S(h)$ for some concrete functions:

<u>Example</u> Determine $\quad S(\chi_{\mathbb{D}}) = ?$

<u>Sol'n</u>: If $g(z) = \begin{cases} \bar z , & |z|<1 \\ \frac{1}{z} , & |z|>1 \end{cases}$ then $g \in \mathring{W}^{1,p}(\mathbb{C})$ for $1<p<\infty$ and $g_{\bar z} = \chi_{\mathbb{D}}$. Thus $S(\chi_{\mathbb{D}}) = g_z = \frac{-1}{z^2} \chi_{\mathbb{C}\setminus\mathbb{D}}$

<u>6.13. Lemma.</u> If

- $|\mu(z)| \le k\,\chi_{B(0,R)}(z)$ for some $R<\infty$, and
- $\varphi \in L^2_0(\mathbb{C})$,

then the equation

(6.8) $\quad \bar\partial \sigma = \mu\, \partial\sigma + \varphi$

has a unique solution $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$ with $\sigma(z) = O(\frac{1}{z})$ at ∞

<u>Proof:</u> By Cor. 6.12, (6.8) $\Leftrightarrow$ $\sigma_{\bar z} = \mu S(\sigma_{\bar z}) + \varphi$
$\Leftrightarrow$ $(I - \mu S)(\sigma_{\bar z}) = \varphi$.

As $\|\mu\|_\infty \|S\|_{L^2\to L^2} \le k\cdot 1 = k < 1$, the usual Neumann series argument shows $(I-\mu S)^{-1} : L^2 \to L^2$.

Thus get

$$\sigma_{\bar z} = (I-\mu S)^{-1}\varphi = \varphi + \mu S(\varphi) + \mu S \mu S(\varphi) + \cdots$$

where the converges in $L^2(\mathbb{C})$-norm; in fact sum $\in L^2_0(\mathbb{C})$

<u>Define</u> now

$$\sigma(z) := \mathcal{C}\left((I-\mu S)^{-1}\varphi\right)$$

By Cor. 6.11 $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$ and $\sigma(z) = \mathcal{C}(\sigma_{\bar z}) = O(1/z)$. why?
Further, $\sigma_{\bar z} = (I-\mu S)^{-1}\varphi \Rightarrow \sigma_{\bar z} - \mu S(\sigma_{\bar z}) = \sigma_{\bar z} - \mu \sigma_z = \varphi$.

<u>Uniqueness</u> If σ_1 another sol'n,

$$(\sigma - \sigma_1)_{\bar z} = \mu(\sigma-\sigma_1)_z \Rightarrow (I-\mu S)\big(\underbrace{(\sigma-\sigma_1)_{\bar z}}_{\in L^2}\big) = 0 \Rightarrow$$

$(\sigma-\sigma_1)_{\bar z} = 0 \Rightarrow \sigma \equiv \sigma_1$ (c.f. p. 58). $\square$

<u>Recall</u> the notation: $\mathring{W}^{1,2}(\mathbb{C}) = \{ g \in W^{1,2}_{loc}(\mathbb{C}) : \nabla g \in L^2(\mathbb{C}) \}$.

<u>6.14. Theorem</u>. If $|\mu(z)| \le k \chi_{B(0,R)}(z)$, $z \in \mathbb{C}$, then there is a unique $f \in W^{1,2}_{loc}(\mathbb{C})$ with

(6.9) $\quad \bar\partial f = \mu(z) \partial f$ a.e. $z \in \mathbb{C}$ & $f(z) = z + O(1/z)$, $|z| \to \infty$.

<u>Proof:</u> Use Lemma 6.13 with $\varphi \equiv \mu \in L^2_0(\mathbb{C})$ and <u>define</u>

$$f(z) := z + \sigma(z) = z + \mathcal{C}\left((I - \mu S)^{-1} \mu \right)$$

by proof of 6.13

$$= z + \mathcal{C}\left(\mu + \mu S \mu + \mu S \mu S \mu + \dots \right)$$

As $(I - \mu S)^{-1} \mu \in L^2_0(\mathbb{C})$, $f(z) = z + O(1/z)$ as $|z| \to \infty$.

Also $f_{\bar z} = \sigma_{\bar z} \underset{\text{Le 6.13}}{=} \mu \sigma_z + \mu = \mu(1 + \sigma_z) = \mu f_z$.

<u>Uniqueness:</u> If f_1 another solution $\Rightarrow$ $\underset{= O(1/z^2)\ (\text{at } \infty)}{\underline{\nabla(f - f_1)}} \in L^2(\mathbb{C})$

and $(I - \mu S)\left((f - f_1)_{\bar z} \right) = (f - f_1)_{\bar z} - \mu (f - f_1)_z = 0$.

As $I - \mu S$ invertible $\Rightarrow$ $(f - f_1)_{\bar z} = 0$ in L^2 $\Rightarrow$ $f - f_1$ analytic (c.f. p. 51)

As $\left. \begin{matrix} f - f_1 = O(1/z) \\ f - f_1 \text{ analytic} \end{matrix} \right\} \Rightarrow f \equiv f_1$. $\square$

<u>6.15. Definition</u>. When $\mathrm{supp}(\mu)$ compact, solutions to (6.9) with $f(z) = z + O(1/z)$ are called <u>principal solutions</u>.

<u>Question</u>: Are principal solutions homeomorphic?

For proving the homeomorphicity we need first:

6.16. Lemma. Let $\mu, \varphi \in C_0^\infty(\mathbb{C})$ and $\sigma_{\bar z} = \mu \sigma_z + \varphi$ as in Lemma 6.13; $|\mu| \le k < 1$ and $\sigma \in \mathring{W}^{1,2}(\mathbb{C})$.

Then $\sigma \in C^\infty(\mathbb{C})$.

Proof: Note first that $\bar\partial S(\varphi) = \bar\partial \partial \mathcal{C}(\varphi) = S(\bar\partial\varphi)$ for any $\varphi \in C_0^\infty(\mathbb{C})$; similarly $\partial S(\varphi) = S(\partial\varphi)$.

On the other hand, proof of Le. 6.13 $\Rightarrow$

$$\sigma_{\bar z} = \varphi + \mu S(\varphi) + \mu S \mu S(\varphi) + \dots = \omega_0 + \omega_1 + \omega_2 + \dots$$

If $D = \alpha\partial + \beta\bar\partial$ (α, β constant) then

$$D\omega_n = D\mu\, S(\mu S \mu \dots \mu S\varphi) + \mu S(D\mu\, S(\mu S \mu \dots \mu S\varphi)) + \dots + \mu S(\mu S(\mu \dots \mu S D\varphi))$$

Thus

$$\| D\omega_n \|_{L^2(\mathbb{C})} \le (n+1) \underbrace{\| S \|_{L^2}^n}_{=1} k^{n-1} \left(\|\varphi\|_{L^2} \|D\mu\|_{L^\infty} + \|D\varphi\|_{L^2} \right)$$

and get

$$h := \sum_{n=0}^\infty D\omega_n \in L^2(\mathbb{C})$$ where sum abs. convergent in the L^2-norm. But then

$$\int_{\mathbb{C}} D\varphi\, \sigma_{\bar z} = - \int_{\mathbb{C}} \varphi h \quad \forall \varphi \in C_0^\infty(\mathbb{C})$$

is $\sigma_{\bar z}$ has L^2-derivatives; $\sigma_{\bar z} \in W^{1,2}(\mathbb{C})$. Can continue this inductively $\Rightarrow$ $\sigma_{\bar z} \in \bigcap_{k=0}^\infty W^{k,2}(\mathbb{C}) \subset C^\infty(\mathbb{C})$. Here the last inclusion uses Fourier - analysis :

$$\phi \in \bigcap_{k=0}^{\infty} W^{k,2}(\mathbb{C}) \iff \int_{\mathbb{C}} (1+|\xi|^2)^k \, |\hat{\phi}(\zeta)|^2 \, dm < \infty \quad \forall\, k \in \mathbb{N},$$

with $f \in C^1(\mathbb{R}^2)$ whenever $|\zeta|\,|\hat{f}(\zeta)| \in L^1(\mathbb{R}^2)$

(or refer to Sobolev embedding).

Finally, $\sigma(z) = \mathcal{C}(\sigma_{\bar{z}}) \in C^\infty(\mathbb{C})$ [Remark 6.6]

$\square$

Now we are ready for:

<u>6.17. Theorem</u> (Measurable Riemann Mapping Th. for $\mu \in C_0^\infty$)

Suppose $\mu \in C_0^\infty(\mathbb{C})$ with $|\mu(z)| \leq k < 1$, $z \in \mathbb{C}$.
Let $f \in W^{1,2}_{loc}(\mathbb{C})$ be the principal solution to

$$(6.10) \qquad \bar{\partial} f = \mu \, \partial f, \quad \text{a.e. } z \in \mathbb{C}.$$

Then

1°) f is a C^∞-diffeomorphism of $\mathbb{C}$ $\Rightarrow$ K-q conformal $\left(K = \frac{1+k}{1-k}\right)$

2°) $J(z, f) > 0 \quad \forall\, z \in \mathbb{C}$.

<u>Proof:</u> $f(z) = z + \mathcal{C}\left((I - \mu S)^{-1} \mu\right) \in C^\infty(\mathbb{C})$ by previous lemma. The main point is to prove 2°). For this

solve the auxiliary equation

$$\sigma_{\bar z} = \mu \sigma_z + \mu_z$$

for $\sigma \in C^\infty(\mathbb{C}) \cap \mathring{W}^{1,2}(\mathbb{C})$, as in Lemma 6.16. Define now

$$F(z) = z + \mathcal{C}(\mu e^\sigma) \tag{6.11}$$

As $\mu e^\sigma \in C_0^\infty(\mathbb{C})$ we have $F \in C^\infty(\mathbb{C})$ with $F(z) = z + O(1/z)$ as $|z| \to \infty$. We claim that F solves (6.10); then F will a principal solution to (6.10), and by uniqueness of such solutions it will follow $f(z) \equiv F(z)$.

For this purpose note first that

$$(\mu e^\sigma)_z = (\mu_z + \mu \sigma_z) e^\sigma = \sigma_{\bar z} e^\sigma = (e^\sigma)_{\bar z}$$

Moreover, then

$$e^{\sigma(z)} - 1 \overset{\text{why?!}}{=} \mathcal{C}((e^\sigma)_{\bar z}) = \mathcal{C}((\mu e^\sigma)_z)$$

$$= \partial \mathcal{C}(\mu e^\sigma) = S(\mu e^\sigma)$$

i.e. we have shown

$$\boxed{e^{\sigma(z)} = 1 + S(\mu e^\sigma)}$$

With this info we can calculate from (6.11) that

$$\bar\partial F = \mu e^\sigma \; ; \quad \partial F = 1 + S(\mu e^\sigma) = e^\sigma$$

Thus $\bar\partial F = \mu \partial F$ and $f \equiv F$! But then

$$J(z,f) = |\partial F|^2 - |\bar\partial F|^2 = |e^{2\sigma}|(1 - |\mu|^2) \geq |e^{2\sigma}|(1-k^2) > 0 \quad \forall z \in \mathbb{C}$$

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To complete 1°), by 2°) and inverse function theorem, f is a local homeomorphism in $\mathbb{C}$. As $f(z) = z + O(1/z)$, f is also a homeo in $\{|z| > R\}$ for some large R; for this consider $h(z) := 1/f(1/z) = \frac{z}{1+O(z)}$ for $|z| < \varepsilon$; as $h'(0) = 1$, h is homeo in a nbhood of origin $\Rightarrow$ f in a nbhood of ∞.

Thus f a local homeo of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the Riemann sphere. In particular f is an open mapping; thus $f\hat{\mathbb{C}}$ open and closed in $\hat{\mathbb{C}}$, hence $f\hat{\mathbb{C}} = \hat{\mathbb{C}}$. To prove injectivity one may use monodromy thm. and the fact that $\hat{\mathbb{C}}$ is simply connected. But there is also an elementary argument: Let

$$A(n) = \{ w \in \hat{\mathbb{C}} : \# f^{-1}(w) = n \} .$$

Then $A(n)$ is open and closed in $\hat{\mathbb{C}}$, thus either $A(n) = \hat{\mathbb{C}}$ or $A(n) = \emptyset$. As $f^{-1}(\infty) = \{\infty\}$ we have $A(1) = \hat{\mathbb{C}}$ and $A(n) = \emptyset$ for $n \neq 1$; in particular every $w \in \hat{\mathbb{C}}$ has only one preimage $\Rightarrow$ f is injective $\Rightarrow$ $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ homeo!

Last, as f is a homeomorphic (and smooth) solution to Beltrami eq'n, f is K-quasiconformal (see p. 43)

$\square$

VI.4. Meas. Riemann Map. Theorem for general μ

For a general $\mu \in L^\infty(\mathbb{C})$, $|\mu| \le k < 1$, we need the so called "good approximation lemma":

<u>6.18. Lemma</u>. Suppose $\{\mu_n\}_{n\ge 1} \subset L^\infty(\mathbb{C})$ are such that $\|\mu_n\|_\infty \le k < 1$ $\forall n \ge 1$, and the limit

$$\mu(z) := \lim_{n\to\infty} \mu_n(z)$$

exists for a.e. $z \in \mathbb{C}$. Suppose further that $f_n : \mathbb{C} \to \mathbb{C}$ are $W^{1,2}_{loc}(\mathbb{C})$-homeos solving

$$\bar\partial f_n = \mu_n(z)\, \partial f_n \quad \text{a.e. } z \in \mathbb{C} \;\&\; f_n(0) = 0,\ f_n(1) = 1.$$

(in part: f_n unique!). Then the limit

$$f(z) := \lim_{n\to\infty} f_n(z)$$

exists $\forall z \in \mathbb{C}$, the convergence is uniform on compact subsets of $\mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$ homeo, $f \in W^{1,2}_{loc}(\mathbb{C})$ and

$$\bar\partial f = \mu(z)\, \partial f \quad \text{a.e. } z \in \mathbb{C}. \tag{6.12}$$

<u>Proof:</u> By previous chapters, f_n are η_K-quasisymm., $K = \frac{1+k}{1-k} < \infty$. Also Exercise set 1 $\Rightarrow$ $\{f_n\}$ equicontinuous, pointwise bdd and there is a subsequence $\{f_{n_k}\}$ such that

$f_{\mu_k}(z) \to f(z)$ unif. on compacts of $\mathbb{C}$, where $f(z)$ is a η_K-quasisymmetric map of $\mathbb{C}$, with $f(0)=0$ and $f(1)=1$. It suffices to show that f solves (6.12), since the normalized solution is unique, and thus all converging subsequences have the same limit!

Let us first show that derivatives of f_{μ_k} converge weakly in $L^2_{loc}(\mathbb{C})$. Indeed, for any $R<\infty$,

$$(6.13)\quad \int_{B(0,R)} |Df_n|^2 \le K \int_{B(0,R)} J(z,f_n) = K\,|f B_n(0,R)|$$

$$\le \pi K \eta(R)^2 ;$$

thus for a further subsequence $\partial f_{\mu_k} \to g$ weakly in $L^2_{loc}(\mathbb{C})$, and similarly $\bar\partial f_{\mu_k} \to h \in L^2_{loc}(\mathbb{C})$.

We claim that $g = \partial f$, $h = \bar\partial f$. Namely, $\forall\, \varphi \in C_0^\infty(\mathbb{C})$

$$\int \partial\varphi\, f = \lim_k \int_{\mathbb{C}} \partial\varphi\, f_{\mu_k} = -\int_{\mathbb{C}} \varphi g \Rightarrow g = \partial f.$$ Same reasoning $\Rightarrow h = \bar\partial f$.

Further, if $\mathrm{supp}(\varphi) \subset B(0,R)$

$$(6.14)\quad \int_{\mathbb{C}} \varphi\,(\bar\partial f_{\mu_k} - \mu\,\partial f_{\mu_k}) = \int_{\mathbb{C}} \varphi\,(\mu_{n_k} - \mu)\,\partial f_{\mu_k}$$

Here L.H.S $\xrightarrow[(k\to\infty)]{} \int_{\mathbb{C}} \varphi\,(\bar\partial f - \mu\,\partial f)$, as the derivatives converge weakly in $L^2(B(0,R))$. On the other hand

R.H.S. of (6.14) $\leq$

$$\|\mathcal{C}(\mu_n-\mu)\|_{L^2}\ \|Df_{n_k}\|_{L^2(B(0,R))} \overset{(6.13)}{\leq}$$

$$\sqrt{\pi K}\,\eta(R)\,\|\phi(\mu_n-\mu)\|_{L^2(\mathbb{C})} \longrightarrow 0$$

Dom. konvergence !

Thus have shown that any limit f satisfies (6.12). $\square$

<u>6.19. Remark</u> The normalization $f_n(0)=0$, $f_n(1)=1$ was used just to get rel. compactness of $\{f_n\}_{n\geq 1}$. If any other normalization gives compactness (and uniqueness), then the proof of Lemma 6.18 works as such in this situation.

<u>6.20. Theorem</u> (M.R.M.T.) If $\mu\in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty \leq k < 1$, then there exists a unique $W^{1,2}_{loc}(\mathbb{C})$-homeo $f:\mathbb{C}\to\mathbb{C}$ with

$$\bar\partial f = \mu\,\partial f \quad \& \quad f(0)=0,\ f(1)=1.$$

<u>Proof</u>: Choose $\mu_n\in C_0^\infty(\mathbb{C})$ with $\mu_n(z)\to\mu(z)$ a.e. z, $\|\mu_n\|_\infty \leq k<1$. Solve then $\bar\partial f_n = \mu_n\,\partial f$ with $f_n:\mathbb{C}\to\mathbb{C}$ homeo (use Thm. 6.17), $f_n(0)=0$, $f_n(1)=1$

[If F_n corresp. principal sol'n, let $f_n(z) = \dfrac{F_n(z) - F_n(0)}{F_n(1) - F_n(0)} = \alpha F_n + \beta$]. Use Lemma 6.18 to get existence and Corollary 6.4 or Remark 6.5 to get uniqueness. $\square$

VII General solutions to Beltrami eq's

In this section we study and classify solutions $g \in W^{1,2}_{loc}(\Omega)$ (or even g with less regularity) to

$$g_{\bar z} = \mu\, g_z \quad , \quad |\mu| \le k < 1 ,$$

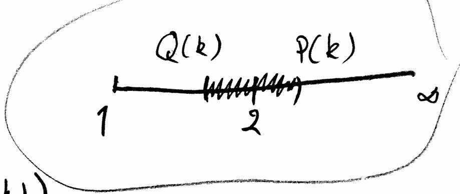
<u>without</u> any further assumptions (such as homeo etc.)

VII.1. L^p-estimates & Caccioppoli Inequalities

Start with further norm bounds for Cauchy & Beurling op.

A. Recall $S : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ bounded with norm $\|S\|_p < \infty$ for $1 < p < \infty$. Also $\|S\|_2 = 1$ and Riesz-Thorin thm $\Rightarrow$ $p \mapsto \|S\|_p$ continuous. Thus there exists exponents

$$(7.1) \qquad Q(k) < 2 < P(k)$$



such that $k\|S\|_p < 1$ whenever $p \in (Q(k), P(k))$.

We call $(Q(k), P(k))$ the critical interval.

Note: as $k \to 0$, $(Q(k), P(k)) \to (1, \infty)$

as $k \to 1$, $(Q(k), P(k)) \to \{2\}$.

If Iwaniec conjecture true $\Rightarrow$ $(Q(k), P(k)) = (1+k, 1+1/k)$.

B. • If $f \in L^p(\mathbb{C})$, $1 < p < 2$, $\Rightarrow$

(7.2) $\mathcal{C} f \in L^r(\mathbb{C})$, $r = \frac{2p}{2-p} > 2$. (Sobolev embedding)

• If $f \in L^2(\mathbb{C}) \Rightarrow \mathcal{C} f \in VMO(\mathbb{C})$; in fact for some $a > 0$,

$$\int_B e^{a|\mathcal{C} f|^2} < \infty \quad \forall \text{ disk } B \subset \mathbb{C}. \quad \text{(Trudinger)}$$

• If $f \in L^p(\mathbb{C})$, $p > 2$ $\Rightarrow$ $\mathcal{C} f \in Lip_\alpha$, $\alpha = 1 - 2/p$

(The proof of the last claim is part of Exercise set 3)

7.1. Example on using the critical interval: If $0 < k < 1$ let $\mathcal{F}_k := \{ f : \mathbb{C} \to \mathbb{C}$ principal soln, $\bar\partial f = \mu \partial f$, $|\mu| \le k \chi_{\mathbb{D}} \}$

We claim that $\mathcal{F}_k$ is compact in topo of unif. convergence on compacts of $\mathbb{C}$. Once this is shown, Thm 6.14, Thm 6.17 and Remark 6.19 show that any principal soln.

(for μ smooth or not) is a homeomorphism!

For compactness of $\mathcal{F}_k$, recall from proof of Thm. 6.14 that $f \in \mathcal{F}_k \Rightarrow$

$$\bar\partial f = (I-\mu S)^{-1}\mu \quad \& \quad f(z) = z + \mathcal{C}\left((I-\mu S)^{-1}\mu\right)$$

Here

$$\|(I-\mu S)^{-1}\mu\|_{L^p(\mathbb{C})} \le C(p)\,\|\mu\|_{L^p(\Delta)} \le \pi^{1/p} C(p),$$

$\forall\, p \in (Q(k), P(k))$. Choosing $2 < p < P(k)$, Exercise 3 $\Rightarrow$

$$|f(z)-f(w)| \le |z-w| + C_1(p)\,\|f_z\|_{L^p(\mathbb{C})}\,|z-w|^{1-2/p} \le |z-w| + C_2\,|z-w|^{1-2/p},$$

for all $f \in \mathcal{F}_k$. Thus $\mathcal{F}_k$ is equicontinuous. But by Thm. 2.10.4 on p. 13A, $\mathcal{F}_k$ is pointwise bdd. Thus Ascoli-Arzela $\Rightarrow$ $\mathcal{F}_k$ compact!

<u>7.2. Theorem</u> (Caccioppoli inequalities). Suppose $0 \le k < 1$, and $f \in W^{1,q}_{loc}(\Omega)$, where $q \in (Q(k), P(k))$. Assume

$$|\bar\partial f(z)| \le k\,|\partial f(z)| \quad \text{a.e. } z \in \Omega. \tag{7.3}$$

Then, if $\eta \in C_0^\infty(\Omega)$, we have

$$\|\eta\, Df\|_{L^s(\mathbb{C})} \le C(s,k)\,\|f\, D\eta\|_{L^s(\mathbb{C})} \quad \forall\, s \in (Q(k), P(k)).$$

Thus $f \in W^{1,s}_{loc}(\Omega)$. In particular, $f \in C(\Omega)$!

<u>Remark</u> $W^{1,p}_{loc}(\Omega) \subset C(\Omega)$ when $\underline{p>2}$ (Exercises 3)
but not when $p \le 2$; take e.g. $f(z) = \log|\log|z||$, $|z|<1$;
$|\nabla f| \simeq \frac{1}{|z||\log|z||} \in L^2_{loc}(\mathbb{D})$.

<u>Proof of Thm 7.2</u>: $(7.3) \Rightarrow f_{\bar z} = \mu f_z$ a.e. $z \in \Omega$,
$|\mu| \le k < 1$. As $\eta \in C_0^\infty(\Omega)$,

$$F := \eta f \in W^{1,q}(\mathbb{C}) \cap L^t(\mathbb{C}),$$

for some $2 < t < P(k)$ [use (7.2) / Sobolev embedding]
Further,

$$F_{\bar z} = \mu F_z + \underbrace{(\eta_{\bar z} - \mu \eta_z) f}_{\equiv h}$$

and so

$$(I - \mu S)(F_{\bar z}) = h$$

[here note: $S(\varphi_{\bar z}) = \varphi_z$ for $\varphi \in W^{1,q}(\mathbb{C})$ globally; thus need η to create a $W^{1,q}(\mathbb{C})$ - fcn out of f.]

Now

(7.4) $$F_{\bar z} = (I - \mu S)^{-1} h \in L^t(\mathbb{C}) \text{ for some } 2 < t < P(k).$$

Thus F (Hölder)-continuous, $F = \mathcal{C}(F_{\bar z}) \in \mathrm{Lip}_{(1-2/t)}$
$\Rightarrow f$ continuous, in fact locally Hölder continuous!

Further, $F_z = S(F_{\bar z}) \in L^t(\mathbb{C})$ and so

(7.5) $$\| DF \|_{L^t(\mathbb{C})} \le \|F_z\|_{L^t} + \|F_{\bar z}\|_{L^t} \le c \|h\|_{L^t} \le c \| f D\eta \|_{L^t(\mathbb{C})}$$

But now h Hölder cont & has compact support $\Rightarrow h \in L^p(\mathbb{C})$ $\forall p \Rightarrow$

Can use (7.4) as long as $t < P(k)$!!

i.e. $F_{\bar z} \in L^s(\mathbb{C})$ for every $s < P(k)$. And (7.5) $\Rightarrow$

(7.6) $$\| DF \|_{L^s(\mathbb{C})} \leq C \| f \, D\eta \|_{L^s(\mathbb{C})} , \qquad s < P(k).$$

Finally, by triangle inequality,

$$\| \eta Df \|_{L^s} = \| DF - f D\eta \|_{L^s(\mathbb{C})} \leq C \| f \, D\eta \|_{L^s(\mathbb{C})} . \square$$

What Caccioppoli inequality gives is an self-improving property for solutions to Beltrami equations: we start with regularity $f \in W^{1,q}_{loc}$ for some $Q(k) < q < 2$ and automatically end with better regularity $f \in W^{1,s}_{loc}$, $2 < s < P(k)$.

But this self-improved regularity is very delicate; it does not work below $Q(k)$:

7.3. Example Let $f(z) = \frac{1}{z} |z|^{1 - 1/K}$. Then

$$|\bar\partial f| = \frac{K-1}{K+1} |\partial f| \quad \& \quad f \in W^{1,q}_{loc}(\mathbb{C}) \quad \forall \, q < 1+k = \frac{2K}{K+1}$$

In fact, $f = \varphi \circ F$, $\varphi(z) = \frac{1}{z}$; $F(z) = z |z|^{\frac{1}{K} - 1}$ is K-qc $(K \geq 1)$.

Remark It can [AIM] be shown that Caccioppoli works for all $q \geq 1+k$.

Next, we need a version of chain rule, that works for quasiregular maps. In this connection it is convenient to introduce the Royden algebra

$$R(\Omega) = C(\Omega) \cap L^\infty(\Omega) \cap \mathring{W}^{1,2}(\Omega)$$

$R(\Omega)$ is indeed an algebra, with product $(fg)(x) = f(x)g(x)$ and norm

$$\| f \|_* = \| f \|_\infty + \| \nabla f \|_{L^2}$$

This makes $R(\Omega)$ a Banach algebra.

We have now the following form of chain rule:

<u>7.4. Theorem</u> Let $f : \Omega \to \Omega'$ be a K-quasiconformal homeo and let $u \in R(\Omega')$. Then $u \circ f \in R(\Omega)$ and

$$\nabla (u \circ f)(z) = Df(z)^t \, \nabla u(fz), \text{ i.e. } D(u\circ f)(z) = Du(fz) \circ Df(z)$$

<u>Proof:</u> see the enclosed copies, p. 76 A–B, and Thm 3.8.2 / AIM. $\square$

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Certainly, we may find simple functions $u_\nu \leqslant u$ converging almost everywhere to u, and for these u_ν we have

$$\int_\Omega u_\nu(f(z))J(z,f) = \int u_\nu(w) \to \int u(w)$$

The only problem we have to overcome then is to show that $u_\nu(f(z)) \to u(f(z))$ almost everywhere. This is clear from Theorem 3.7.5. □

The most convenient way to present the chain rule is in terms of the Royden algebra $R(\Omega)$ of a domain Ω. Recall that the elements of this algebra are the continuous and bounded functions with distributional derivatives in $L^2(\Omega)$,

$$R(\Omega) = C(\Omega) \cap L^\infty(\Omega) \cap \mathbb{W}^{1,2}(\Omega)$$

Here, and throughout this monograph, we use the notation

$$\mathbb{W}^{1,p}(\Omega) = \{v : \nabla v \in L^p(\Omega, \mathbb{C}^2)\}$$

for the homogeneous Sobolev space, that is, for the space of (locally integrable) functions with L^p-integrable gradient. No assumptions are made here on the L^p-integrability of the function itself.

For a bounded domain, however, $R(\Omega) \subset W^{1,2}(\Omega)$. The Royden algebra is a Banach algebra with the norm

$$\|v\|_* = \|v\|_\infty + \|\nabla v\|_2$$

According to the next result, quasiconformal mappings of Ω preserve the Royden algebra $R(\Omega)$. In fact, the reader may easily verify that a quasiconformal $f : \Omega \to \Omega'$ induces an algebra isomorphism $T_f : R(\Omega') \to R(\Omega)$, $T_f(v) = v \circ f$. Somewhat deeper lies the fact that every algebra isomorphism of the Royden algebra is of this type [230].

Theorem 3.8.2. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal mapping and let $u \in R(\Omega')$. Then the composition $v = u \circ f$ lies in the Royden algebra $R(\Omega)$ with derivative*

$$\nabla v(z) = D^t f(z) \nabla u(f(z)), \quad \text{almost everywhere in } \Omega \tag{3.39}$$

Further, we have the estimate

$$\int_\Omega |\nabla v(z)|^2 \, dz \leqslant K \int_{\Omega'} |\nabla u(z)|^2 \, dz \tag{3.40}$$

Proof. The main point is showing that $u \circ f$ has square-integrable distributional derivatives, and the argument is a variation of Lemma 3.5.2. We first assume that $u_\beta \in C^\infty(\overline{\Omega'})$ converge to u locally uniformly and in $\mathbb{W}^{1,2}(\Omega)$. From Lemma 3.5.2 we have $u_\beta \circ f \in \mathbb{W}^{1,2}(\Omega)$ and at the points of differentiability of f, hence almost everywhere,

$$\nabla(u_\beta \circ f)(z) = D^t f(z) \nabla u_\beta(f(z)),$$

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which we may write equivalently in the integral form

$$-\int_{\Omega} \nabla\varphi(z)\, u_\beta(f(z)) = \int_{\Omega} \varphi(z) D^t f(z) \nabla u_\beta(f(z)) \tag{3.41}$$

for every test funtion $\varphi \in C_0^\infty(\Omega)$.

Here passing to the limit as $\beta \to \infty$ on the left hand side poses no difficulty whatsoever as $u_\beta(f(z)) \to u(f(z))$ locally uniformly. However, we must justify passing to the limit on the right hand side. We do this as follows, using the fact that f is K-quasiconformal to give $|D^t f|^2 \leqslant KJ(z,f)$.

$$\begin{aligned}
&\int_{\Omega} |D^t f(z)\nabla u_\beta(f(z)) - D^t f(z) \nabla u(f(z))|^2 \\
&= \int_{\Omega} |D^t f(z)[\nabla u_\beta - \nabla u](f(z))|^2 \\
&\leqslant K \int_{\Omega} J(z,f)|[\nabla u_\beta - \nabla u](f(z))|^2 \\
&= K \int_{\Omega'} |\nabla u_\beta - \nabla u|^2
\end{aligned}$$

by the change-of-variable formula (3.38). Now,

$$\int_{\Omega'} |\nabla u_\beta - \nabla u|^2 \to 0 \quad \text{as } \beta \to \infty$$

We end up with the identity

$$-\int_{\Omega} \nabla\varphi(z)\, u(f(z)) = \int_{\Omega} \varphi(z) D^t f(z) \nabla u(f(z)) \tag{3.42}$$

for every $C_0^\infty(\Omega)$-test function. Directly from the definition of the Sobolev space we see that $v(z) = u(f(z))$ lies in the algebra $W^{1,2}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, and its gradient is as given at (3.39). The integral inequality is immediate from the change-of-variable formula of Theorem 3.8.1. □

3.9 Quasisymmetry and Equicontinuity

Next, we set about proving Theorem 3.1.3. We first establish the result for quasisymmetric mappings, which will in turn establish the desired result. First, let us prove that a family of normalized quasisymmetric mappings is equicontinuous. We shall use this fact a bit later on in the book. We draw the reader's attention to Section 2.9.4 where the definition of equicontinuity is given.

Theorem 3.9.1. *Let $A \subset \mathbb{C}$ with $0, 1 \in A$. Then the family of all η-quasisymmetric maps $f : A \to \mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$ is equicontinuous.*

Proof. Let $a_0 \in A$. Then for any $a \in A$,

$$\frac{|f(a_0) - f(a)|}{|f(a_0) - f(1)|} \leqslant \eta\left(\frac{|a_0 - a|}{|a_0 - 1|}\right) \quad \text{and} \quad \frac{|f(a_0) - f(a)|}{|f(a_0) - f(0)|} \leqslant \eta\left(\frac{|a_0 - a|}{|a_0|}\right)$$

With these tools we can classify all $W^{1,2}_{loc}$-solutions to the Beltrami eq.

<u>7.5. Theorem</u> (Stoilow Factorization) Let $h \in W^{1,2}_{loc}(\Omega)$ be (some) solution to

$$(7.7) \qquad \bar\partial h = \mu(z)\, \partial h \quad \text{a.e. } z \in \Omega; \quad |\mu(z)| \le k < 1 \quad \text{a.e. } z.$$

Then, if $f : \Omega \to \Omega'$ is a $W^{1,2}_{loc}$-homeo solving (7.7), we have

$$h(z) = \Phi \circ f(z) \quad , \; z \in \Omega,$$

where Φ <u>analytic</u> in $\Omega' = f(\Omega)$.

<u>Proof:</u> Define $\Phi = h \circ f^{-1}$. Then h and thus Φ continuous by Thm 7.2. By restricting to a subdomain have $\Phi \in L^\infty(\Omega)$ and $h \in R(\Omega) \overset{\text{Thm 7.4}}{\Rightarrow} \Phi \in R(\Omega)$. As Thm 7.4 gives pointwise chain rule, $D\Phi = Dh(f^{-1}) \circ Df^{-1}$, i.e.

$$\bar\partial \Phi(f_z) = h_z \cdot (f^{-1})_{\bar z}(f) + h_{\bar z} \cdot \overline{(f^{-1})_z(f)}$$

But the chain rule formulae from p 52 (with $g = f^{-1}$) give

$$(f^{-1})_{\bar z}(f) = -\frac{f_{\bar z}(z)}{J(z,f)} \overset{(7.7)}{=} -\frac{\mu(z) f_z(z)}{J(z,f)}$$

and

$$\overline{(f^{-1})_z}(f) = \frac{f_z(z)}{J(z,f)}$$

$$\Rightarrow \bar\partial \Phi(f(z)) \cdot J(z,f) = -h_z\, \mu(z) f_z + h_{\bar z}(z) f_z(z) \overset{(7.7)}{=} 0 \quad \text{a.e.}$$

As the qconf f has Lusin properties by Thm. 4.1, it follows $\bar\partial\Phi = 0$ a.e., and as $\Phi \in R(\Omega') \subset W^{1,2}_{loc}(\Omega')$, Weyl's lemma proves Φ analytic. $\square$

<u>Remarks</u> a) M.R.M.T gives homeomorphic solutions to (7.7), just set $\mu_0(z) = \begin{cases} \mu(z), & z \in \Omega \\ 0, & z \in \mathbb{C}\setminus\Omega \end{cases}$, let F be the $W^{1,2}_{loc}$-homeo sol'n to $\bar\partial F = \mu_0 \partial F$, and take $f = F|_\Omega$.

b) If Ω simply connected, let $\varphi : \Omega' = f(\Omega) \to \Omega$ be a conformal map. Then $\tilde f = \varphi \circ f$ solves the <u>same</u> Beltrami equation (7.7). Thus can <u>always</u> find a homeomorphic $W^{1,2}_{loc}(\Omega)$-solution to (7.7) with

$$f : \Omega \to \Omega.$$

c) Conversely to Thm 7.5., if Φ analytic in $\Omega' = f(\Omega)$ then $h := \Phi \circ f$ solves (7.7). Thus

$$\boxed{h \text{ a } W^{1,2}_{loc}\text{-sol. to (7.7)} \iff h = \Phi \circ f, \quad \begin{matrix} f \text{ q conf} \\ \Phi \text{ analytic} \end{matrix}}$$

classifies all solution to the Beltrami eq'n.

$$|h_{\bar z}| \le k|h_z| \quad \text{or} \quad h_{\bar z} = \tilde\mu h_z, \quad |\tilde\mu| \le k.$$

By Stoilow, $h = \Phi \circ f$, where f is K-quasiconformal (recall (5.2)/p. 43), and where Φ analytic. Thus $u = \operatorname{Re} h = \underbrace{(\operatorname{Re}\Phi)}_{\equiv w \text{ harmonic}} \circ f$. $\square$

VIII Holomorphic Motions

Holomorphic motions give yet another approach — completely different from the previous ones — to quasiconformal mappings.

Let us first return to the proof of M.R.M.T. with $|\mu| \le k\,\chi_{B(0,R)}$, for some $R < \infty$.
If f is the principal solution to $\bar\partial f = \mu \partial f$, then

$$f(z) = z + \mathcal{C}\big((I-\mu S)^{-1}\mu\big) \qquad (8.1)$$ (c.f. p. 62)

which shows that f depends holomorphically on μ!
This can be interpreted in many ways; we use the following:

Define $\quad \mu_\lambda^{(z)} = \frac{\lambda}{k}\mu(z) \quad , \quad |\lambda| < 1.$

Then $\|\mu_\lambda\|_\infty \le |\lambda| < 1$ and μ_λ has compact support $\Rightarrow$ Can solve

$$(8.2) \qquad \bar\partial f = \mu_\lambda \partial f \quad ; \quad f = f^\lambda(z) = z + O(1/z) \text{ at } \infty.$$

To make better use of representation (8.1) here, we apply the L^p-theory:

Let $\varrho < 1$ and fix $2 < p < P(\varrho)$. i.e. $\varrho \|S\|_p < 1$

Then

$$|\lambda| < \varrho \Rightarrow \|\mu_\lambda\|_\infty \|S\|_p \le \varrho \|S\|_p < 1 \quad \Rightarrow$$

$\sum_{n=0}^{\infty} (\mu_\lambda S)^n \mu_\lambda \in L^p(\mathbb{C})$, where sum converges absolutely,

$\sum \|(\mu_\lambda S)^n \mu_\lambda\|_{L^p} < \infty$. For each fixed z, $\frac{1}{\zeta - z} \in L^q(B(0,R))$, $(q<2!)$

$\frac{1}{p} + \frac{1}{q} = 1$ and therefore, for any $z \in \mathbb{C}$,

$$f^\lambda(z) = z + \mathcal{C}\Big(\sum_{n=0}^{\infty} (\mu_\lambda S)^n \mu_\lambda\Big)(z)$$

$$= z + \sum_{n=0}^{\infty} \mathcal{C}\big((\mu_\lambda S)^n \mu_\lambda\big)(z) = z + \sum_{n=0}^{\infty} \lambda^n \mathcal{C}\Big[\big(\tfrac{\mu}{k} S\big)^n \tfrac{\mu}{k}\Big](z)$$

(one can use also Exercise set 3 to get this representation).

Thus we see that for any fixed $z \in \mathbb{C}$, $f^\lambda(z)$ can be represented as a power series in λ, thus $\lambda \mapsto f^\lambda(z)$ is analytic in λ, $|\lambda| < \varrho$. But $\varrho < 1$ arbitrary $\Rightarrow$ get

<u>8.1. Theorem</u> Let f be the principal solution to $\bar\partial f = \mu \partial f$ and let f^λ solve (8.2) with $\mu_\lambda = \frac{\lambda}{k}\mu$. Then

a) $f^k = f$

b) $\lambda \mapsto f^\lambda(z)$ analytic in $\mathbb{D}$, for any fixed $z \in \mathbb{C}$.

c) $z \mapsto f^\lambda(z)$ homeo in $\mathbb{C}$, for any fixed λ.

d) $f^0(z) \equiv z$

<u>Proof</u> a); d) follow from uniqueness of principal sol's and b) was explained above & c) on p. 71 – 72. $\square$

Thus we have embedded any (principal) quasiconf. map $f: \mathbb{C} \to \mathbb{C}$ to a holomorphic flow of qc maps!

<u>Remark</u> Often principal solution is not the best normalization. If we set

$$(8.3) \qquad F^\lambda(z) = \frac{f^\lambda(z) - f^\lambda(0)}{f^\lambda(1) - f^\lambda(0)} \quad \Rightarrow \quad F^\lambda(0)=0, \ F^\lambda(1)=1$$

and a) – d) still hold for F^λ ($F^k = F$, the normalized solution to $\bar\partial F = \mu \partial F$) !!

<u>Question</u> Can we take the limit $R \to \infty$ and keep all good properties a) – d) of Theorem 8.1 ?