Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 4
18.2.2016

1. Let $I = [0,1]$ be the unit interval. Given a parameter $0 < p < 1$, we define a 4-adic coding of I as follows: First, we partition $I = I_1^{(p)} \cup I_2^{(p)} \cup I_3^{(p)} \cup I_4^{(p)}$ by subdividing I at the points

$$0 < p(1-p) < p < 1 - p(1-p) < 1.$$

The intervals $I^{(p)}_{i_1 i_2 i_3,\dots,i_n}$ are defined inductively, so that if $L^{(p)}_{i_1,i_2,\dots,i_n}$ is an affine map that takes $I^{(p)}_{i_1 i_2 i_3,\dots,i_n}$ to I, then L maps partition points to partition points.

For two parameters $0 < p, q < 1$, there exists a unique homeomorphism $H_{p,q}$ of $[0,1]$ which maps $I^{(p)}_{i_1 i_2 i_3,\dots,i_n}$ onto $I^{(q)}_{i_1 i_2 i_3,\dots,i_n}$.

(a) Show that $H_{p,q}$ is quasisymmetric.

(b) Show that if $p \neq q$, then $H_{p,q}$ is differentiable at no point of I. Conclude that $H_{p,q}$ is not absolutely continuous.

Remark. It may seem more natural to split $I = [0,p] \cup [p,1]$ in just two pieces. This construction; however, does not lead to a quasisymmetries.

2. According to Theorem 3.11 in the notes, if $f : \Omega \to \Omega'$ is a η-quasisymmetric map between bounded domains in $\mathbb{R}^2$, then

$$L_f(z_0) := \limsup_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

is in L^2_{loc}, which is slightly better than the weak-L^2_{loc} estimate for

$$L^\epsilon_f(z_0) := \sup_{|z - z_0| \le \epsilon} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

(a) Show that in 1-dimension, the analogous function $L_f(x) \in L^1$.

(b) It would appear that $\int_a^b L^\epsilon_f \to \int_a^b L_f$ converge as $\epsilon \to 0$ (since the functions L^ϵ_f are decreasing in ϵ), suggesting that f is absolutely continuous. (This seems to contradict the previous exercise.) Does the monotone convergence not apply?