Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 3
11.2.2016

1. Let $\alpha > 0$. Compute the dilatations $\mu_L = \overline{\partial} L / \partial L$ and $\mu_R = \overline{\partial} R / \partial R$ where

$$L(x + iy) = (\alpha + 1)x + iy$$

is the linear stretch mapping and

$$R(z) = z|z|^\alpha$$

is radial stretch mapping. Verify explicitly that $\mu_L = (e^z)^* \mu_R$.

2. Write down a formula for the Jacobian of a map $f : \mathbb{C} \to \mathbb{C}$ in terms of the complex derivatives ∂f and $\overline{\partial} f$.

3. Construct an example of a map $f : E \to \mathbb{R}^2$, with $E \subset \mathbb{R}^2$, that is weakly quasisymmetric but not quasisymmetric.

4. Suppose $u \in C^2(\overline{\Omega})$ where $\Omega = \{z : r < |z| < 1\}$, with boundary values $u(re^{i\theta}) = 0$, $u(e^{i\theta}) = 1$, $0 \le \theta \le 2\pi$. Find the optimal lower bound for the energy

$$\mathscr{E}(u) = \int_\Omega |\nabla u|^2 dxdy.$$

5. Suppose $u : \Omega' \to \mathbb{R}$ is a Lipschitz function. Given $f \in W^{1,p}_{\text{loc}}$ with $f(\Omega) \subseteq \Omega'$, show that $u \circ f \in W^{1,p}_{\text{loc}}$.

A word on notation. Above,

$$\partial f := \frac{\partial f}{\partial z}, \quad \overline{\partial} f := \frac{\partial f}{\partial \overline{z}}.$$

If f is holomorphic, then

$$f^*\left(\mu(z)\frac{d\overline{z}}{dz}\right) = \mu(f(z))\frac{\overline{f'(z)}}{f'(z)} \cdot \frac{d\overline{z}}{dz}.$$