Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 2
4.2.2016

1. Show that any quasisymmetric map $f : \mathbb{C} \to \mathbb{C}$ is surjective.

2. Let $f(z) = |z|^\alpha$, $z \in \mathbb{C}$ and $\alpha > -1$. If $g(z) = (\alpha/2)\overline{z}|z|^{\alpha-2}$, show that
$$\int_{\mathbb{C}} f(z)\partial\phi(z)dm(z) = -\int_{\mathbb{C}} g(z)\phi(z)dm(z)$$
for every $\phi \in C_0^\infty(\mathbb{C})$. Conclude that the weak derivative $\partial f = g$.

   Hint. Use Green's formula
$$\int_{\partial D} hd\overline{z} = -\frac{1}{2i}\int_D \partial h dm, \qquad h \in C^\infty(\overline{D}),$$
with $D = D_\epsilon = \{z : \epsilon < |z| < R\}$ and take $\epsilon \to 0$.

   A word on notation. As is standard, we use the abbreviation $\partial f := \frac{\partial f}{\partial z}$.

3. Given a function $f : \mathbb{R} \to \mathbb{R}$, let
$$L_f^\epsilon(z) = \sup_{|h|<\epsilon} \frac{|f(z+h) - f(z)|}{h}.$$
   (a) Show that $L_f^\epsilon \in L^1_{\text{loc}}$ implies that f is absolutely continuous.
   (b) State the analogues of Lemmas 3.8 and 3.9 from the lecture notes in the 1-dimensional case. In particular, decide if the arguments imply that "all quasisymmetric maps of the real line are absolutely continuous."

4. The *Lebesgue density theorem* states that for a set $E \subset \mathbb{R}$, and almost any point $x \in E$,
$$\frac{m(E \cap B(x,r))}{m(B(x,r))} \to 1, \qquad \text{as } r \to 0.$$
This is a special case of the *Lebesgue differentiation theorem* which says that for a function $f \in L^1$,
$$\lim_{r\to 0} \frac{\int_{B(x,r)} f(y)dm(y)}{B(x,r)} \to f(x).$$
The aim of this exercise is to prove the above theorems.

(a) Observe that the Lebesgue differentiation theorem is valid for continuous functions.

Consider the maximal function

$$Mh(x) = \sup_{r>0} \frac{\int_{B(x,r)} h(y)dm(y)}{B(x,r)}.$$

One useful fact is the estimate

$$m(\{Mh > t\}) < C/t \cdot \|h\|_{L^1}.$$

(You are not asked to prove this estimate here.)

(b) Suppose $\epsilon > 0$. Using an approximation of an L^1 function by continuous functions $g_n \to f$, and $h_n = f - g_n$, show that the set of points x where

$$\limsup_{r\to 0} \frac{\int_{B(x,r)} f(y)dm(y)}{B(x,r)} > f(x) + \epsilon,$$

has measure 0.

(c) Similarly, show that the set of points x where

$$\liminf_{r\to 0} \frac{\int_{B(x,r)} f(y)dm(y)}{B(x,r)} < f(x) - \epsilon,$$

has measure 0.