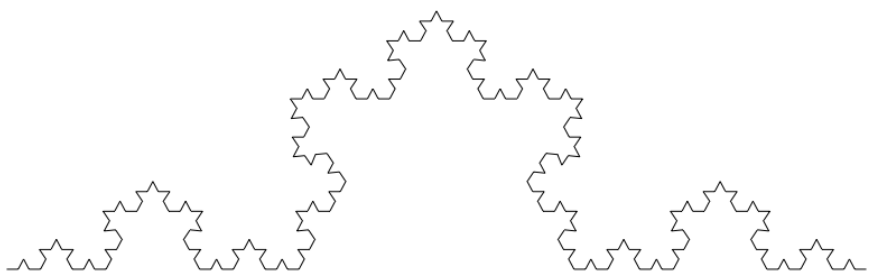
Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 1
28.1.2016

1. For $\alpha, \beta > 0$, let $f : \mathbb{R} \to \mathbb{R}$ that $f(x) = x^\alpha$ if $x > 0$ and $f(x) = -|x|^\beta$ for $x < 0$. Show that f is quasisymmetric if and only if $\alpha = \beta$.

   (Note that f and f^{-1} are Hölder continuous mappings for any $\alpha, \beta > 0$.)

2. (a) Show that the snowflake K may be represented as the image of the line segment $[0, 1]$ under a Ct^α-quasisymmetric map $f : [0, 1] \to K$ for some $\alpha < 1$.

   (b) Nevertheless, show that one cannot take α to be arbitrarily close to 1.



3. Suppose $f : \mathbb{D} \to \mathbb{C}$ is a conformal mapping that admits an η-quasisymmetric extension to the plane. For an arc $I \subset \mathbb{S}^1$, define its conformal midpoint z_I as the midpoint of the hyperbolic geodesic joining z_1 and z_2. Show that there exists a constant C (depending on η) so that

$$\frac{1}{C} \cdot |f'(z_I)| \le \frac{\operatorname{diam} f(I)}{\operatorname{diam} I} \le C \cdot |f'(z_I)|$$

   (Hint: use Koebe's distortion theorem.)

4. Let

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

   be a conformal map of the exterior unit disk $\{z : |z| > 1\}$ to a domain $\Omega = \mathbb{C} \setminus K$. The aim of this problem is to find a formula for the area of K in terms of the coefficients b_k.

(a) Find an asymptotic formula (as $R \to \infty$) for the area $A(R)$ of the compact set enclosed by the curve $f(S_R)$, where $S_R = \{z : |z| = R\}$.

(b) Compute

$$B(R) = \lim_{\rho \to 1} \int_{\rho < |z| < R} |f'(z)|^2.$$

by using the power series expansion.

(c) Analyze $\lim_{R \to \infty}(A(R) - B(R))$.