Department of Mathematics and Statistics, University of Helsinki
Numerical methods and the C language, Winter and Spring 2016

Workshop 12

Exercise 12 FILE: h12.tex printed on November 20, 2015at 11.09.

1. By using the program `odeint` included in e.g. `myod1.c` on the www-page, solve the second-order constant coefficient differential equation $y'' + ay' + by = g(x)$ initial value problem $y(0) = 0$, $y'(0) = 1$ when $a = 2$, $b = 3$, $g(x) = exp(-x)$ and compare to the exact solution $0.5*r+r*(c_1*\cos(x/c_2)+(c_2)*\sin(x/c_2))$, $r = \exp(-x)$, $c_1 = -0.5$, $c_2 = pow(2.0, -0.5)$. *Hint:* The situation can be reduced to solving the system of equations
$$\begin{cases} \frac{dy_1}{dx} = y_2(x), \\ \frac{dy_2}{dx} = g(x) - ay_2(x) - by_1(x) \end{cases}$$

2. The so called *Lorentz system of differential equations* occur in e.g.weather forecasting:
$$\frac{dy_1}{dx} = s(y_2(x) - y_1(x)),$$
$$\frac{dy_2}{dx} = ry_1(x) - y_2(x) - y_1(x)y_3(x),$$
$$\frac{dy_3}{dx} = y_1(x)y_2(x) - by_3(x).$$
By using `rkdumb` or `odeint`, solve this in the interval $[0, 8]$ when $s = 10,\ r = 128,\ b = 8$ and the initial values are $7.7, -15.6, 90.4$.

3. Modify the program `myod1.cpp` as to draw pictures of the solution function of the following initial value problem. The hypergeometric function $F(a, b; c; x)$ satisfies
$$\begin{aligned} \frac{dF(a,b;c;x)}{dx} &= \frac{ab}{c}F(a+1, b+1; c+1; x) \quad \text{and} \\ F(a,b;c;0) &= 1. \end{aligned}$$
For $x \in (0, 1)$ and $a, b \in (0, 1)$
$$\begin{aligned} \frac{dF(a-1,b;c;x)}{dx} &= \frac{(a-1)}{x}[F(a,b;c;x) - F(a-1,b;c;x)], \\ \frac{dF(a,b;c;x)}{dx} &= \frac{(c-a)F(a-1,b;c;x) + (a-c+bx)F(a,b;c;x)}{x(1-x)}. \end{aligned}$$
Hence the IVP
$$\begin{aligned} \frac{dy1}{dx} &= \frac{a-1}{x}[y2(x) - y1(x)], \\ \frac{dy2}{dx} &= \frac{(c-a)y1(x) + (a-c+bx)y2(x)}{x(1-x)}, \\ y1(x1) &= F(a-1,b;c;x1), \\ y2(x1) &= F(a,b;c;x1) \end{aligned}$$
has solution $y1 = F(a-1, b; c; x), y2 = F(a, b; c; x)$. Set concrete values such as $x1 = 0.1, a = 1.2, b = 2.3, c = 3.5$ and compute the numerical and the exact solution on $[0.1, 0.9]$. Also compute the difference between these.

4. Use the program `odeint.c` to compute the integral

$$\int_0^x 10\sin(2^n \pi t)dt, n = 1, 2, ..., x \in (0, 1),$$

and print the difference to the exact value.

5. We consider for $h > 0$ a quadrature formula of the form

$$\int_0^{3h} f(x)dx = w_1 f(0) + w_2 f(2h) + w_3 f(3h)$$

where the weights $w_j, j = 1, 2, 3$, are chosen so that this holds as equality for each of the three functions $1, x, x^2$.

(a) Find the coefficients $w_j, j = 1, 2, 3$.

(b) Show that the formula also holds as equality for $f(x) = ax^2 + bx + c$.

(c) What is the error if $h = 1, f(x) = 1 + x + x^2 + x^3$?