## Department of Mathematics and Statistics, University of Helsinki
## Numerical methods and the C language, Winter and Spring 2016

## Workshop 4

Exercise 4 FILE: h04.tex printed on November 20, 2015at 11.04.

1. Derive the normal equations for the coefficients $a, b, c$ in fitting the parabola

$$a * x * x + b * x + c$$

to the set of datapoints $(x[i], y[i]),\ i = 1, \ldots, n.$

*Hint:* By setting

$$S = \sum_{i=1}^{n} (a * x[i]^2 + b * x[i] + c - y[i])^2$$

the normal equations are

$$\frac{\partial S}{\partial a} = 0, \quad \frac{\partial S}{\partial b} = 0, \quad \frac{\partial S}{\partial c} = 0.$$

Note that these equations are linear with respect to $a, b$ and $c$, and can therefore be solved by `LUsolve`.

2. Modify the program `myjgsit.cpp` (see the www-page) and use the Gauss-Seidel method to solve a $m \times m$ system $Ax = b$ with a diagonally dominating matrix $A$. Also solve with the algorithm `LUsolve` (www-page). In both cases compute the residual. Print the results in the form

```
n   resid/GS   resid/LUsolve

5
10
15
...
50
```

where $n$ is the number of iterations in the Gauss-Seidel algorithm. Use matrices and vectors with random entries.

3. Maritime Rescue Service receives several messages from fishermen about an oil ship which seems to require quick help. The analysis of the message $k$ suggests that the ship is located on the line $y = c_k x + d_k, k = 1, 2, 3, 4,$ where $c = (-0.4, 0.6, -1.3, 2.0)$ $d = (0.5, 0.0, 1.2, -0.8)$. (a) Use the measurements 1,2 to determine the location. (b) Use the measurements 1,2,3 to determine the location. (c) Use the measurements 1,2,3,4 to determine the location. For (b)-(c) the system is overdetermined, use the LSQ method to find the solution.

Hint: Modify the program `myang2.cpp` on the www-page.

4. Write a collection of subroutines to carry out the following operations: (a) compute the p-norm of each column of an $m \times n$ matrix, (b) given an $m \times n$ matrix $a$ generate another $m \times n$ matrix $b$ obtained from $a$ by multiplying each column by a constant so that each column of $b$ has norm 1, (c) given an $m \times n$ matrix $a$ with each column having norm 1, generate another $n \times n$ matrix $b$ such that $b_{i,j}$ is the scalar product of the columns $i$ and $j$ of $a$.

5. For an $n \times n$ matrix $A = (a_{i,j})$ let $P_i = \sum_{j=1, j \neq i}^{n} |a_{i,j}|$. Gershgorin's theorem says that the eigenvalues of a complex $n \times n$ matrix $A$ lie in the union of the disks $\{z \in C : |z - a_{i,i}| < P_i\}$ $i = 1, ..., n$. Verify Gershgorin's theorem experimentally for real random matrices. Use the following format to print the eigenvalues, nearest diagonal elements etc.

```
lambda   Nearest a[i][i]   i    |lamda-a[i][i]|   P_i
```

[An analytic proof of is given here (not part of this problem). Gershgorin's theorem follows from the theorem of Lévy-Desplanques according to which the diagonal dominance $|a_{i,i}| > P_i$ for all $i = 1, ..., n$ implies that $\det(A) \neq 0$ for complex $n \times n$ matrices $A$. To prove the Lévy-Desplanques theorem (Marcus- Minc, ISBN 0-486-67102-X, p. 146) suppose that $\det(A) = 0$. Then the system $Ax = 0$ has a solution $x = (x_1, ..., x_n)$ with at least one $x_i \neq 0$. Choose $x_r$ with $|x_r| \geq |x_i|$ for all $i = 1, ..., n$. Then $|a_{r,r}||x_r| = |-\sum_{j \neq r} a_{r,j} x_j| \leq \sum_{j \neq r} |a_{r,j}||x_j| \leq |x_r| P_r$ which contradicts the diagonal dominance condition. Next, to prove Gershgorin's theorem let $\lambda_i$ be a an eigenvalue of $A$. Then $\det(\lambda_i I - A) = 0$ and, by the Lévy-Desplanques theorem, $|\lambda_i - a_{i,i}| < P_i$ at least for one $i$.]