Department of Mathematics and Statistics, University of Helsinki
Numerical methods and the C language, Winter and Spring 2016

Workshop 3
Exercise 3 FILE: h03.tex printed on November 20, 2015at 11.03.

1. Make functions which generate random upper and lower triangular matrices and functions which solve an upper and lower triangular system of equations, $Ux = b$ and $Lx = b$ respectively. These solvers (`usolve` and `lsolve`) should take as an argument an upper or respectively a lower triangular matrix as well a constant vector b. Solve the systems for random matrices and for a randomly generated vector b.

2. For random numbers $c_0, \ldots, c_4$ and fixed $a, b \in \mathbb{R}, a < b$, compute the value of the integral

$$I(a,b) = \int_a^b \sum_{j=0}^{4} c_j x^j \, dx$$

(a) analytically, (b) numerically with the following trapezoidal formula. Let $f(x) = \sum_{j=0}^{4} c_j x^j$, $n \in \mathbb{N}$, $h = (b-a)/n$ and $x_k = a + kh, k = 0, ..., n$. Then

$$I(a,b) \approx I(a,b,n) = h \sum_{k=1}^{n} \left[\frac{1}{2} f(x_{k-1}) + \frac{1}{2} f(x_k) \right] = h \sum_{k=0}^{n} f(x_k) - \frac{h}{2}(f(x_0) + f(x_n)).$$

Print the results for $n = 10, 100, \ldots$ in the form

```
  n     I(a,b,n)  |I(a,b,n) - I(a,b)|     h*h

  10       ...              ....              ...
 100       ...              ....              ...
1000       ...              ....              ...    .
```

3. Is the diagonal dominance of a square matrix preserved under the multiplication of two such matrices? Is the inverse of a diagonally dominating matrix diagonally dominating? Is the inverse of a tridiagonal matrix tridiagonal? Remember that a square $n \times n$ matrix $A = (a_{ij})$ is diagonally dominating if $|a_{i,i}| > \sum_{j=1, j \neq i}^{n} |a_{i,j}|$ for all $i = 1, \ldots, n$ and tridiagonal if $a_{i,j} = 0$ for $|i - j| > 1$.

4. At the youthful age of 103 years L. Vietoris (1891-2002) proved in 1994 the following result (Notices of AMS Nov. 2002).

Theorem. Let $a_0 \geq a_1 \geq ... \geq a_n > 0$. If $a_{2k} \leq \frac{2k-1}{2k} a_{2k-1}$ for $1 \leq k \leq \frac{n}{2}$, then for all $t \in (0, \pi)$

$$f_1(t) \equiv \sum_{k=1}^{n} a_k \sin kt > 0\,, \text{ and } f_2(t) \equiv \sum_{k=0}^{n} a_k \cos kt > 0.$$

Verify these inequalities by generating random sequences of the coefficients satisfying these constraints and by graphing the functions f_1, f_2.

5. For real $n \times n$ matrices A with eigenvalues λ_i show that the following results hold

$$\mathrm{tr}(A) \equiv \sum_{i=1}^{n} a_{i,i} = \sum_{i=1}^{n} \lambda_i, \qquad \det(A) = \prod_{i=1}^{n} \lambda_i.$$

   Use the program `myeigen.cpp` (www-page/Chapter 11) to verify this experimentally. If you are using GSL, it is sufficient to study symmetric matrices only.