UH Introduction to mathematical finance I, Exercise-6 (02.03.2016)

In all the exercises we consider random variables defined on a probability space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$.

1. Let be $\tau(\omega) \in \mathbb{N}$ and $\sigma(\omega) \in \mathbb{N}$ stopping times in the filtration $\mathbb{F}$. Show that $\tau \wedge \sigma$ is also a stopping time. Is their sum $\tau(\omega) + \sigma(\omega)$ a stopping time?

   Solution: By definition $\tau(\omega)$ is a stopping if $\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t$, but

   $$\{\omega : \tau(\omega) \wedge \sigma(\omega) \le t\} = \{\omega : \tau(\omega) \le t\} \cap \{\omega : \sigma(\omega) \le t\} \in \mathcal{F}_t.$$

   Also the sum is a stopping time because

   $$\{\omega : \tau(\omega) + \sigma(\omega) \le t\} = \cup_{s=1}^{t} [\{\omega : \tau(\omega) \le s\} \cup \{\omega : \sigma(\omega) \le t - s] \in \mathcal{F}_t$$

2. Consider the positive sequence $Z_t > 0$ for $t \in \mathbb{N}$. Show that

   $$Z_t^{-1} = Z_0^{-1} - \sum_{u=1}^{t} (Z_u Z_{u-1})^{-1} \Delta Z_u$$

   Solution: We proceed by induction: the claim is trivial for $t = 1$ and if the assume the claim to be true up to $t - 1$, then we have

   $$Z_0^{-1} - \sum_{u=1}^{t} (Z_u Z_{u-1})^{-1} \Delta Z_u = Z_{t-1}^{-1} - \frac{Z_t - Z_{t-1}}{Z_{t-1} Z_t} = Z_t^{-1}.$$

3. In the probability space $(\Omega, \mathcal{F}, P)$, let be $Q \ll P$ and $Z = dQ/dP \in L^1(\Omega, \mathcal{F}, P)$. Show that $P \ll Q$ if and only if $Z(\omega) > 0$ P a.s. and then

   $$\frac{dP}{dQ}(\omega) = Z^{-1}(\omega)$$

   Solution: First, let us assume that $Q \sim P$ and we show that there exists $dQ/dP = Z(\omega) > 0$ P almst surely. Since $Q \ll P$, the Radon-Nikodym theorem gaurantees that there exists a random variable $Z(\omega) \in L^1(\Omega, \mathcal{F}, P)$ such that $Q(A) = \int_A Z dP$ for any $A \in \mathcal{F}$. We just need to show that $Z > 0$ almost surely. To to do this, let be $B = \{\omega : Z(\omega) = 0\}$, then we have

   $$Q(B) = \int_B Z(\omega) dP = 0,$$

   but since $P \ll Q$, then we know that $P(B) = 0$.

   For the other direction, we assume that $Q \ll P$ and that there $Z = dQ/dP \in L^1(\Omega, \mathcal{F}, P)$ and $Z > 0$ almost surely. This implies that for any $A \in \mathcal{F}$

   $$Q(A) = \int_A Z dP.$$

Now we wnat to show that if $Q(A) = 0$, then $P(A)$, which menas that $P \ll Q$. In thact, if $Q(A) = 0$, then $0 = \int_A ZdP$. But $Z > 0$ almost surely, then $P(A) = 0$.

Therefore, finally

$$dQ = ZdP \Longleftrightarrow dP = Z^{-1}dQ.$$

4. Let be $f(x)$ a continuos function with a continuous derivative and X_t a discrete time process.

   (a) Show the discrete Ito lemma:

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \sum_{s=1}^{t}(f(X_s) - f(X_{s-1}) - f'(X_{s-1})\Delta X_s) \\ &= \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \frac{1}{2}\sum_{s=1}^{t} \Delta f'(X_s)\Delta X_s + R(f', X, t) \end{aligned}$$

   where

$$R(f', X, t) = \sum_{s=1}^{t} \int_{X_{s-1}}^{X_s} \left(f'(u) - \frac{f'(X_{s-1}) + f'(X_s)}{2}\right) du \qquad (0.1)$$

   Solution: Note that

$$R(f', X, t) = \sum_{s=1}^{t} \left(\Delta f'(X_s) - \frac{\Delta X_s}{2}(f'(X_{s-1}) + f'(X_s))\right),$$

   then by rearranging the terms

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \sum_{s=1}^{t}(\Delta f(X_s) - f'(X_{s-1})\Delta X_s) \\ &= \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \sum_{s=1}^{t} \frac{\Delta X_s}{2}(f'(X_s) - f'(X_{s-1})) + R(f', X, t) \\ &= \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \frac{1}{2}\sum_{s=1}^{t} \Delta f'(X_s)\Delta X_s + R(f', X, t) \end{aligned}$$

   (b) Show that, if X_t is a $\mathbb{F}$-martingale and f is convex and has bounded derivative, then $f(X_t)$ is a submartingale.

   Solution: Let be $|f'(x)| \leq K$, then, since X_t is a martingale, $X_t \in L^1$ and thus $\mathbb{E}|f(X_t)| \leq \mathbb{E}|f(X_0)| + CKt(1 + \sup_{u \leq t} \mathbb{E}|X_u|) < \infty$. Moreover,

$$\begin{aligned} \mathbb{E}[f(X_t)|\mathcal{F}_{t-1}] &= f(X_{t-1}) + \frac{1}{2}\mathbb{E}[\Delta f'(X_t)\Delta X_t|\mathcal{F}_{t-1}] + \mathbb{E}[\Delta f'(X_t)|\mathcal{F}_{t-1}] - \mathbb{E}[\frac{\Delta X_t}{2} f'(X_t)|\mathcal{F}_{t-1}] \\ &= f(X_{t-1}) + \frac{1}{2}\mathbb{E}[f'(X_{t-1})\Delta X_t|\mathcal{F}_{t-1}] + \mathbb{E}[\Delta f'(X_t)|\mathcal{F}_{t-1}] \\ &= f(X_{t-1}) + \mathbb{E}[\Delta f'(X_t)|\mathcal{F}_{t-1}] \geq f(X_{t-1}) \end{aligned}$$

where in the last line we used the convexity of f. We note that a more direct way of proving the claim is by using the Jensen inequality:

$$\mathbb{E}[f(X_t)|\mathcal{F}_{t-1}] \geq f(\mathbb{E}[X_t|\mathcal{F}_{t-1}]) = f(X_t).$$

(c) Show that if f'' is α-Hölder continuous, i.e. if there exists $\alpha \in (0,1]$ and $C > 0$ such that $|f''(x) - f''(y)| \leq C|x-y|^\alpha$, then

$$|R(f', X, t)| \leq const \sum_{s=1}^{t} |\Delta X_s|^2 \max\{|\Delta X_s|^\alpha : 1 \leq s \leq t\}$$

Solution: We will use the trapezoidal rule which says that

$$\int_a^b dx\, |g(x) - \frac{1}{2}(g(a)+g(b))| \leq C|b-a|^2 \sup_{y_1,y_2 \in (a,b)} |g'(y_1) - g'(y_2)|$$

To show this formula, let us consider the approximating polynomial $P(x)$ defined as

$$P(x) = -\frac{x-b}{h} g(a) + \frac{x-a}{h} g(b)$$

where $h = b - a$, then one has by the mean value theorem

$$g(x) - P(x) = \frac{x-b}{h}(g(a) - g(x)) + \frac{x-a}{h}(g(x) - g(b)) = \frac{(x-b)(x-a)}{h}(g'(x_1) - g'(x_2))$$

where $x_1 \in (a, x)$ and $x_2 \in (x, b)$. This implies that

$$\int_a^b dx |g(x) - P(x)| \leq h^2 \sup_{y_1,y_2 \in (a,b)} |g'(y_1) - g'(y_2)|$$

Using the representation (0.1), the trapezoidal formula and the Hölder inequality we immediately have

$$\begin{aligned} |R(f', X, t)| &\leq C \sum_{s=1}^{t} |\Delta X_s|^2 \sup_{x,y \in (X_{s-1}, X_s)} |f''(x) - f''(y)| \\ &\leq C \max_{s \leq t}\{|\Delta X_s|^\alpha\} \sum_{s=1}^{t} |\Delta X_s|^2. \end{aligned}$$

5. Let be $X_t = \sum_{s=1}^t \Delta X_s$ a random path process where $\Delta X_s \in \{-1, +1\}$ and $f(x) = |x - x_0|$ with $x_0 \in \mathbb{R}$. Then

$$f'(x) = \text{sign}(x - x_0) = \begin{cases} 1 & \text{if } x > x_0 \\ 0 & \text{if } x = x_0 \\ -1 & \text{if } x < x_0 \end{cases}$$

Define

$$L_t^{x_0} = \sum_{s=1}^{t} 1(X_{s-1} = x_0).$$

Show the discrete Takana's lemma:

$$|X_t - x_0| = L_t^{x_0} + \sum_{s=1}^{t} \mathrm{sign}(X_t - x_0)\Delta X_s.$$

Solution: Let us use the decomposition

$$f(X_t) - f(X_0) = \sum_{s=1}^{t} f'(X_{s-1})\Delta X_s + \sum_{s=1}^{t}[f(X_s) - f(X_{s-1}) - f'(X_{s-1})\Delta X_s] \tag{0.2}$$

so that

$$|X_t - x_0| = \sum_{s=1}^{t} \mathrm{sign}(X_{s-1} - x_0)\Delta X_s + \sum_{s=1}^{t}[|X_s - x_0| - |X_{s-1} - x_0| - \mathrm{sign}(X_{s-1} - x_0)\Delta X_s]. \tag{0.3}$$

Let us now look at the second term and split its summands as follows:

$$\begin{aligned}
&[|X_s - x_0| - |X_{s-1} - x_0| - \mathrm{sign}(X_{s-1} - x_0)\Delta X_s]1(X_{s-1} = x_0) \qquad (0.4)\\
&+ [|X_s - x_0| - |X_{s-1} - x_0| - \mathrm{sign}(X_{s-1} - x_0)\Delta X_s]1(X_{s-1} > x_0)\\
&+ [|X_s - x_0| - |X_{s-1} - x_0| - \mathrm{sign}(X_{s-1} - x_0)\Delta X_s]1(X_{s-1} < x_0)\\
&= |\Delta X_s|1(X_{s-1} = x_0)\\
&+ (X_s - x_0 - X_{s-1} + x_0 - \Delta X_s)1(X_{s-1} > x_0)\\
&+ (-X_s + x_0 + X_{s-1} - x_0 + \Delta X_s)1(X_{s-1} < x_0)\\
&= |\Delta X_s|1(X_{s-1} = x_0) = 1(X_{s-1} = x_0),
\end{aligned}$$

therefore we have

$$|X_t - x_0| = \sum_{s=1}^{t} \mathrm{sign}(X_t - x_0)\Delta X_s + \sum_{s=1}^{t} 1(X_{s-1} = x_0).$$