Kompleksianalyysi I kurssin jatkokurssi: Model solutions 4

1. Exercise 3.2 (b) and Exercise 3.5 (b).

   Solutions 1. (3.2(b)) We seek the Laurent series at the origin of the function $f(z) = 1/\sin(z)$. We examine the function $g : \mathbb{D}(0,\pi) \to \mathbb{C}$

$$g(z) = \begin{cases} \frac{1}{\sin(z)} - \frac{1}{z}, & \text{when } z \neq 0, \\ 0, & \text{when } z = 0. \end{cases}$$

   As in Exercise 3.2(a) we use l'Hospital's rule and get that g is continuous and

$$\lim_{z\to 0} \frac{\dfrac{1}{\sin(z)} - \dfrac{1}{z}}{z} = \frac{1}{6}.$$

   In other words, the function g has a derivative in the entire disk $\mathbb{D}(0,\pi)$. Especially g is analytic in the disk. We can therefore write g as the Taylor series

$$g(z) = \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots,$$

   which converges when $|z| < \pi$. Now we easily find the wanted Laurent series:

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots.$$

   The series converges when $0 < |z| < \pi$.

   (3.5 (b)) We calculate the residue of the function

$$f(z) = \frac{\exp(z)}{\sin^2(z)}$$

   in all its poles.

   The poles of order 2 are found in $z = m\pi$, where $m \in \mathbb{Z}$. The residue calculation formula gives

$$
\begin{aligned}
\operatorname{Res}(f, z=m\pi) &= \lim_{z\to m\pi} \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\left((z-m\pi)^2 \frac{\exp(z)}{\sin^2(z)}\right) \\
&= \lim_{z\to m\pi} \Big(\frac{\exp(z)(z-m\pi)^2\sin(z)}{\sin^3(z)} + \frac{\exp(z)2(z-m\pi)\sin(z)}{\sin^3(z)} \\
&\quad - \frac{\exp(z)2(z-m\pi)^2\cos(z)}{\sin^3(z)}\Big) \quad (\text{vi betecknar } z-m\pi = u) \\
&= \lim_{u\to 0}\left(\exp(u+m\pi)\frac{u^2\sin(u)+2u\sin(u)-2u^2\cos(u)}{\sin^3(u)}\right) \\
&= \exp(m\pi)\underbrace{\lim_{u\to 0}\left(\frac{u^2\sin(u)+2u\sin(u)-2u^2\cos(u)}{\sin^3(u)}\right)}_{\mathrm{A}}.
\end{aligned}
$$

We examine the limit A using l'Hospital's rule:

$$
\begin{aligned}
\mathrm{A} &= \lim_{u\to 0}\frac{u^3}{\sin^3(u)}\left(\frac{u^2\sin(u)+2u\sin(u)-2u^2\cos(u)}{u^3}\right) \\
&= 1\cdot\lim_{u\to 0}\left(\frac{\frac{\mathrm{d}}{\mathrm{d}u}(u^2\sin(u)+2u\sin(u)-2u^2\cos(u))}{\frac{\mathrm{d}}{\mathrm{d}u}u^3}\right) \\
&= \cdots = 1.
\end{aligned}
$$

Finally we get

$$\operatorname{Res}(f, z=m\pi) = \exp(m\pi).$$

2. Determine the integral

$$\int_0^{2\pi}\frac{\mathrm{d}t}{13+12\cos(t)}.$$

Solutions 2. We remember that

$$\cos(t) = \frac{1}{2}\left(\exp(it)+\exp(-it)\right).$$

Next we denote $z=\exp(it)$, and get

$$\mathrm{d}z = i\exp(it)\mathrm{d}t \implies \mathrm{d}t = \frac{\mathrm{d}z}{iz}.$$

From the substitution follows, that

$$\frac{1}{13+12\cos(t)} = \frac{1}{13+6\left(z+\dfrac{1}{z}\right)} = \frac{z}{13z+6z^2+6},$$

and we get

$$\int_0^{2\pi} \frac{\mathrm{d}t}{13+12\cos(t)} = -i\int_{|z|=1} \frac{\mathrm{d}z}{6z^2+13z+6}.$$

We would like to use the Residue Theorem, so we examine the function

$$f(z) = \frac{1}{6z^2+13z+6} = \frac{1}{6\left(z+\dfrac{2}{3}\right)\left(z+\dfrac{3}{2}\right)}.$$

The function has simple poles in $z=-\frac{2}{3}$ and $z=-\frac{3}{2}$, but only the first of these is found inside the path of integration, $\partial\mathbb{D}(0,1)$. We calculate the residue at the point:

$$\operatorname{Res}\left(f(z);-\frac{2}{3}\right) = \lim_{z\to-\frac{2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{6\left(z+\dfrac{2}{3}\right)\left(z+\dfrac{3}{2}\right)} = \frac{1}{5}.$$

Now, using the Residue Theorem, we get

$$-i\int_{|z|=1}\frac{\mathrm{d}z}{6z^2+13z+6} = -i\cdot 2\pi i\cdot\operatorname{Res}\left(f(z);-\frac{2}{3}\right) = \frac{2\pi}{5}.$$

The value of the first integral is therefore $-\frac{2\pi}{5}$.

3. Determine the integral

$$\int_0^\infty \frac{x\sin(x)}{(x^2+1)^2}\mathrm{d}x.$$

Solutions 3. We notice first, that

$$g(x) = \frac{x\sin(x)}{(x^2+1)^2}$$

is even for all $x \in \mathbb{R}$, i.e. $g(x) = g(-x)$. Now we can write our integral on the form

$$\begin{aligned}
I = \int_0^\infty \frac{x\sin(x)}{(x^2+1)^2}\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin(x)}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\frac{1}{2i}(\exp(ix)-\exp(-ix))}{(x^2+1)^2}\mathrm{d}x \\
&= \underbrace{\frac{1}{4i}\int_{-\infty}^{\infty} \frac{x\exp(ix)}{(x^2+1)^2}\mathrm{d}x}_{A} - \underbrace{\frac{1}{4i}\int_{-\infty}^{\infty} \frac{x\exp(-ix)}{(x^2+1)^2}\mathrm{d}x}_{B}.
\end{aligned}$$

Integral A: We form a path consisting of the real interval $[-R, R]$ and the half circle $\{z \in \mathbb{C} : |z| = R, \mathrm{Im}(z) > 0\}$ in the upper half plane. Here R is some positive real number, and we travel in the positive direction. If we then denote the path along $[-R, R]$ by $\gamma_{[-R,R]}$, and the path along the arc of the half circle by γ_R, we can form the integral

$$\begin{aligned}
I' &= \frac{1}{4i}\int_{\gamma_{[-R,R]}*\gamma_R} \frac{z\exp(iz)}{(z^2+1)^2} \\
&= \frac{1}{4i}\int_{\gamma_{[-R,R]}} \frac{z\exp(iz)}{(z^2+1)^2}\mathrm{d}z + \frac{1}{4i}\int_{\gamma_R} \frac{z\exp(iz)}{(z^2+1)^2}\mathrm{d}z.
\end{aligned}$$

Jordan's Lemma gives that the second of these integrals has the value 0, so we have

$$I' = \frac{1}{4i}\int_{\gamma_{[-R,R]}} \frac{z\exp(iz)}{(z^2+1)^2}\mathrm{d}z + 0 = \frac{1}{4i}\int_{-R}^{R} \frac{z\exp(iz)}{(z^2+1)^2}\mathrm{d}z.$$

We can calculate the value of the integral I' using the Residue Theorem. The function

$$f(z) = \frac{z\exp(iz)}{(z^2+1)^2}$$

has a pole of order 2 in $z = i$, and it is the only pole inside the area restricted by $\gamma_{[-R,R]} * \gamma_R$. We calculate the residue in the point:

$$\mathrm{Res}(f; i) = \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}(z-i)^2 \frac{z\exp(iz)}{(z-i)^2(z+i)^2} = \frac{1}{4e}.$$

By the Residue Theorem we have

$$I' = \frac{1}{4i} \cdot 2\pi i \mathrm{Res}(f; i) = \frac{2\pi i}{4i} \cdot \frac{1}{4e} = \frac{\pi}{8e}.$$

Since $A = \lim_{R\to\infty} I'$ we get $A = \pi/(8e)$.

Integral B: Similarly we calculate the value of the integral B, now along the arc of the halfcircle in the lower half plane. Finally we get $B = -\pi/(8e)$. From this follows that the value of the wanted integral is

$$I = A - B = \frac{\pi}{8e} + \frac{\pi}{8e} = \frac{\pi}{4e}.$$

4. (1) Let Q be a square (a cube in the plane), with corner points in

$$(N+1/2)(1+i), (N+1/2)(-1+i)$$
$$(N+1/2)(-1-i), (N+1/2)(1-i),$$

$N \in \mathbb{Z}$ fixed. Show that along the sides of the cube $|\cot(\pi z)| < A$, where A is constant.

(2) Let f be analytic in the plane, apart from a limited number of poles z_j, $j = 1, 2, \dots l$. Assume that these poles are not integer points.

If the inequality $|f(z)| \leq M/|z|^k$, where $k > 1$ and M are constants independent of N, holds at the border points z of the cube, show then that

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{l} \mathrm{Res}(\pi \cot(\pi z) f(z); z_j),$$

where z_j is a pole of the function f.

Solutions 4. (1) We denote $z = x + iy$, where $x, y \in \mathbb{R}$. We examine the square of the wanted expression:

$$K^2 := |\cot(\pi z)|^2 = \left|\frac{\cos(\pi z)}{\sin(\pi z)}\right|^2 = \left|\frac{\cos(\pi(x+iy))}{\sin(\pi(x+iy))}\right|^2 = \left|\frac{\cos(\pi x + i\pi y)}{\sin(\pi x + i\pi y)}\right|^2$$
$$= \left|\frac{\cos(\pi x)\cos(i\pi y) - \sin(\pi x)\sin(i\pi y)}{\sin(\pi x)\cos(i\pi y) + \sin(i\pi y)\cos(\pi x)}\right|^2$$
$$= \left|\frac{\cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y)}{\sin(\pi x)\cosh(\pi y) + i\sinh(\pi y)\cos(\pi x)}\right|^2$$

Since $|\omega|^2 = \omega \cdot \overline{\omega}$ we get

$$K^2 = \frac{\cos^2(\pi x)\cosh^2(\pi y) + \sin^2(\pi x)\sinh^2(\pi y)}{\sin^2(\pi x)\cosh^2(\pi y) + \sinh^2(\pi y)\cos^2(\pi x)}$$
$$= \frac{\cos^2(\pi x)(1+\sinh^2(\pi y)) + \sin^2(\pi x)\sinh^2(\pi y)}{\sin^2(\pi x)(1+\sinh^2(\pi y)) + \cos^2(\pi x)\sinh^2(\pi y)}$$
$$= \frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\sin^2(\pi x) + \sinh^2(\pi y)}.$$

Now we have along the vertical sides$x = \pm(N + 1/2)$ we have:

$$\cos^2(\pi x) = 0 \quad \text{and} \quad \sin^2(\pi x) = 1,$$

and therefore

$$K^2 = \frac{\sinh^2(\pi y)}{1 + \sinh^2(\pi y)} \leq 1 \implies |\cot(\pi z)| \leq 1.$$

Along the horizontal sides $y = \pm(N + 1/2)$ we get:

$$K^2 \leq \frac{1 + \sinh^2(\pi y)}{\sinh^2(\pi y)} = \frac{1}{\sinh^2(\pi y)} + 1 < \frac{1}{\sinh^2(\pi/2)} + 1 < \frac{6}{5}.$$

We get $|\cot(\pi z)| < \sqrt{6/5}$.

(2) By the Residue Theorem we have

$$\frac{1}{2\pi i}\int_{\partial Q_N} \pi\cot(\pi z) f(z)\mathrm{d}z = \sum_{s=-N}^{N} \mathrm{Res}(\pi\cot(\pi z)f(z); s) + \sum_{s=1}^{l} \mathrm{Res}(\pi\cot(\pi z)f(z); z_s).$$

The function $\cot(\pi z)$ has a simple point at every integer point. We calculate the residue in these points:

$$\mathrm{Res}(\pi\cot(\pi z)f(z); s) = f(s).$$

We notice, that

$$\left|\frac{1}{2\pi i}\int_{\partial Q} \pi\cot(\pi z)f(z)\mathrm{d}z\right| \leq \int_{\partial Q} |\cot(\pi z)||f(z)|\mathrm{d}z$$
$$\leq \sqrt{\frac{6}{5}}\frac{M}{N^k} \to 0, \text{ when } N \to \infty.$$

It follows, that

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{l} \mathrm{Res}(\pi\cot(\pi z) f(z); z_j).$$

5. Show, that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a),$$

where $a > 0$.

Solutions 5. We denote

$$f(z) = \frac{1}{z^2+a^2}, \quad a > 0.$$

Now, by to Exercise 4.4, we get

$$\begin{aligned} S := \sum_{n=-\infty}^{\infty} f(n) &= \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} \\ &= -\sum_{j=1}^{2} \mathrm{Res}\left(\pi\cot(\pi z)\frac{1}{z^2+a^2}; z_j = (-1)^{j+1}ai\right) \\ &= -\mathrm{Res}\left(\pi\cot(\pi z)\frac{1}{z^2+a^2}; ai\right) \\ &\quad - \mathrm{Res}\left(\pi\cot(\pi z)\frac{1}{z^2+a^2}; -ai\right), \end{aligned}$$

where $z = \pm ai$ are first order poles of the function $\pi\cot(\pi z) f(z)$. We get

$$\mathrm{Res}(\pi\cot(\pi z) f(z); ai) = \frac{\pi\cot(\pi ai)}{2ai},$$

and similarly

$$\mathrm{Res}(\pi\cot(\pi z) f(z); -ai) = \frac{-\pi\cot(-\pi ai)}{2ai}.$$

Now we have

$$S = -\left(\frac{\pi}{2ai}\cot(\pi ai) - \frac{\pi}{2ai}\cot(-\pi ai)\right).$$

Using the identities

$$\begin{aligned}
\sin(z) &= \frac{1}{2i}(\exp(iz) - \exp(-iz)), \\
\cos(z) &= \frac{1}{2}(\exp(iz) + \exp(-iz)), \\
\cot(z) &= \frac{\cos(z)}{\sin(z)}, \\
\coth(z) &= \frac{\exp(z) + \exp(-z)}{\exp(z) - \exp(-z)},
\end{aligned}$$

we get, that

$$\cot(\pi a i) - \cot(-\pi a i) = i \coth(-\pi a) - i \coth(\pi a).$$

Finally we get

$$S = -\frac{\pi}{2ai}\left(i \coth(-\pi a) - i \coth(\pi a)\right) = \frac{\pi}{a} \coth(\pi a).$$