Kompleksianalyysi I kurssin jatkokurssi: Model solutions 3

1. Find the Laurent series of the function $f(z) = \frac{1}{z^2(z-3)^2}$ at the center $z = 3$ and determine the area of convergence. What type of singularity is the point $z = 3$?

   Solutions 1. By the Binomial Theorem we have:

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots, \quad |z| < 1,$$

   and it follows, that

$$\left(1+\frac{u}{3}\right)^{-2} = 1 + (-2)\frac{u}{3} + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \cdots, \quad \left|\frac{u}{3}\right| < 1. \tag{1}$$

   Using (**??**) and the change of variable $z = u + 3$ we get

$$\begin{aligned}\frac{1}{z^2(z-3)^2} &= \frac{1}{(3+u)^2u^2} = \frac{1}{9u^2\left(1+\frac{u}{3}\right)^2} \\ &= \frac{1}{9u^2}\left(1 + \frac{-2u}{3} + \frac{(-2)(-3)u^2}{2!\cdot 3^2} + \frac{(-2)(-3)(-4)u^3}{3!\cdot 3^3} + \cdots\right) \\ &= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots \\ &= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4}{243}(z-3) + \cdots.\end{aligned}$$

   It follows directly, that $z = 3$ is a pole of order 2. The series converges $\forall z$, such that $0 < |z-3| < 3$.

   Alternatively we can notice, that

$$\frac{1}{z^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z} = -\frac{1}{3}\frac{d}{dz}\frac{1}{1-\left(-\frac{z-3}{3}\right)},$$

   and use the geometric series.

2. (1) Determine the type of singularity of the function $f : \mathbb{C}\setminus\{j\pi : j \in \mathbb{Z}\}$, $f(z) = \frac{1}{\sin z} - \frac{1}{z}$ at the origin.

   (2) Find the Laurent series of the function $f(z) = \frac{1}{\sin z}$ at the origin and determine the area of convergence.

Solutions 2. (1) We modify the following expression:

$$f(z) = \frac{1}{\sin z} - \frac{1}{z} = \frac{z - \sin(z)}{z \sin(z)}.$$

When $z \to 0$, both the numerator and the denominator tends to zero. We can use l'Hospital's rule:

$$\lim_{z\to 0} \frac{z - \sin(z)}{z\sin(z)} = \lim_{z\to 0} \frac{\frac{d}{dz}(z - \sin(z))}{\frac{d}{dz}(z\sin(z))} = \lim_{z\to 0} \frac{1 - \cos(z)}{\sin(z) + z\cos(z)}.$$

Once again we can use l'Hospital's rule, and finally we get

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin(z)}{\cos(z) + \cos(z) - z\sin(z)} = 0.$$

Since the limit exists and is finite, we see that $z = 0$ is a removable singularity.

(2) Moved to Exercise Session 4.

3. Let f and g be analytic in the disk $\mathbb{D}(z_0, 1)$ and let g have a simple zero at the point z_0. Show, that

$$\operatorname{Res}\left(\frac{f(z)}{g(z)}; z_0\right) = \frac{f(z)}{g'(z)}.$$

Solutions 3. We ca give the functions f and g as power series:

$$f(z) = \sum_{k\geq 0} a_k (z - z_0)^k$$
$$g(z) = \sum_{k\geq 1} b_k (z - z_0)^k = (z - z_0) \sum_{k\geq 1} b_k (z - z_0)^{k-1}.$$

We set $h(z) := \sum_{k\geq 1} b_k (z - z_0)^{k-1}$, and note, that $h(z_0) \neq 0$. Now

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)h(z)},$$

where f/h is analytic. The Taylor expansion gives us

$$\operatorname{Res}\left(\frac{f(z)}{g(z)}; z_0\right) = \frac{a_0}{b_1} = \frac{f(z_0)}{g'(z_0)}.$$

Alternatively: The function g has a simple zero at z_0, i.e. the function f/g has a simple pole at z_0. Using the residue calculation formula we get:

$$\begin{aligned}\operatorname{Res}\left(\frac{f(z)}{g(z)}; z_0\right) &= \lim_{z\to z_0}\frac{(z-z_0)f(z)}{g(z)} = \lim_{z\to z_0}\frac{(z-z_0)f(z)}{g(z)-0}\\ &= \lim_{z\to z_0}\frac{f(z)}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{f(z_0)}{g'(z_0)}.\end{aligned}$$

The last equality is true, since the derivative of the analytic function g exists in $z = z_0$.

4. Calculate the residue of the function

$$f(z) = \frac{z^2+2}{(\exp(z)-1)\cos(z)}$$

in the point $z_0 = 2\pi i$.

Solutions 4. Using the result in Exercise 3 we define

$$f(z) = \frac{z^2+2}{\cos(z)} \quad \text{ja} \quad g(z) = \exp(z) - 1,$$

and get

$$\operatorname{Res}\left(\frac{z^2+2}{(\exp(z)-1)\cos(z)}; 2\pi i\right) = \frac{\frac{(2\pi i)^2+2}{\cos(2\pi i)}}{\exp(2\pi i)} = \frac{-4\pi^2+2}{\cosh(2\pi)}.$$

5. Determine the residues of the functions

$$(a) \quad g(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}, \quad (b) \quad f(z) = \frac{\exp(z)}{\sin^2(z)}$$

at all their poles in the complex plane $\mathbb{C}$.

Solutions 5. (a) The poles of the function are $z = -1$, $z = 2i$ and $z = -2i$. For the calculation of the residue at the pole z_0 we have the formula

$$\operatorname{Res}(f; z_0) = \lim_{z\to z_0}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z)),$$

where k is the order. We notice, that $z = -1$ is a pole of order 2, so according to the formula we get

$$\begin{aligned}\operatorname{Res}(g(z); -1) &= \lim_{z\to -1} \frac{d}{dz}\frac{z^2-2z}{z^2+4} \\ &= \lim_{z\to -1} \frac{(2z-2)(z^2+4)-2z(z^2-2z)}{(z^2+4)^2} = -\frac{14}{25}.\end{aligned}$$

At the points $z = 2i$ and $z = -2i$ the poles are simple, and by direct calculation we get

$$\begin{aligned}\operatorname{Res}(g(z), 2i) &= \frac{7}{25} + \frac{i}{25}, \\ \operatorname{Res}(g(z), -2i) &= \frac{7}{25} - \frac{i}{25}.\end{aligned}$$

(b) Moved to Exercise Session 4.