Kompleksianalyysi I kurssin jatkokurssi: Model solutions 2

1. Let $a, b \in \mathbb{C}$. Determine the factors of the Laurent series of the function

$$f(z) = \exp(az + bz^{-1}) \text{ ja } g(z) = \sin(a(z + z^{-1})).$$

centred at the origin.

Solutions 1. We first examine the function $f(z) = \exp(az + b/z)$ in the annulus $D = \{z : 0 < |z| < \infty\}$. According to the Laurent Theorem the factors are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{(z)^{n+1}} \mathrm{d}z.$$

We fix n:

$$\begin{aligned} \frac{f(z)}{z^{n+1}} &= \frac{\exp\left(\frac{b}{z}\right)\exp(za)}{z^{n+1}} = \frac{1}{z^{n+1}} \exp\left(\frac{b}{z}\right) \sum_{k=0}^{\infty} \frac{z^k a^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{z^{k-n-1} a^k}{k!} \exp\left(\frac{b}{z}\right). \end{aligned}$$

From this follows, that

$$\begin{aligned} \int_{|z|=1} \frac{f(z)}{z^{n+1}} \mathrm{d}z &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{|z|=1} a^k z^{k-n-1} \exp\left(\frac{b}{z}\right) \mathrm{d}z \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\int_{|z|=1} a^k z^{k-n-1} \sum_{j=0}^{\infty} \frac{b^j}{j! z^j} \mathrm{d}z}_{\mathrm{I}}. \end{aligned}$$

We notice that

$$\begin{aligned} I &= \int_{|z|=1} \left(a_k z^{k-n-1} \left(1 + \frac{b}{z} + \frac{b^2}{2z^2} + \cdots \right) \right) \mathrm{d}z \\ &= \int_{|z|=1} a_k z^{k-n-1} \mathrm{d}z + \int_{|z|=1} a_k z^{k-n-1} \frac{b}{z} \mathrm{d}z + \cdots, \end{aligned}$$

and since $\int_{|z|=1}(1/z)\mathrm{d}z = 2\pi i$ and $\int_{|z|=1} h(z) = 0$, when h is analytic, we get

$$I = \begin{cases} 0, & k-n<0 \\ \dfrac{2\pi i}{(k-n)!}a^k b^{k-n}, & k-n \geq 0 \end{cases}.$$

From this follows, that

$$a_n = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{1}{k!}I = \sum_{k=\max\{0,n\}}^{\infty}\frac{a^k b^{k-n}}{k!(k-n)!}.$$

Next we examine the function $g(z) = \sin(a(z+z^{-1}))$ in the same annulus. We remember, that

$$\sin(a(z+z^{-1})) = \frac{1}{2i}(\exp(ia(z+z^{-1})) - \exp(-ia(z+z^{-1}))),$$

and (based on the first part) we get

$$\begin{aligned} a_n &= \frac{1}{2i}\left(\sum_{k=\max\{0,n\}}^{\infty}\frac{(ia)^{k+k-n}}{k!(k-n)!} - \sum_{k=\max\{0,n\}}^{\infty}\frac{(-ia)^{2k-n}}{k!(n-k)!}\right) \\ &= \sum_{k=0}^{\infty}\frac{(ia)^{2k+|n|}}{2ik!(|n|+k)!}. \end{aligned}$$

2. Find the Laurent series of the functions $f(z) = z^{-1}$ and $g(z) = z^{-2}$ in the annulus $\{z : 1 < |z-i| < \infty\}$.

Solutions 2. We begin by examining the function f. We modify the expression z^{-1}:

$$\frac{1}{z} = \frac{1}{i+(z-i)} = \frac{1}{i}\frac{1}{1-i(z-i)}.$$

We have $|i(z-i)| > 1$ when $z \in \{z : 1 < |z-i| < \infty\}$, and using the geometric series we get:

$$\frac{1}{z} = \frac{1}{z-i}\frac{1}{1-\dfrac{1}{i(z-i)}} = \frac{1}{z-i}\sum_{n=0}^{\infty}\frac{1}{i^n(z-i)^n}$$
$$= \frac{1}{z-i}\sum_{n=-\infty}^{0} i^n(z-i)^n = \sum_{n=-\infty}^{0} i^n(z-i)^{n-1}$$
$$= \sum_{n=-\infty}^{-1} i^{n+1}(z-i)^n,$$

which is the Laurent series in the annulus.

Next we examine the function g. We notice, that

$$g(z) = z^{-2} = -f'(z),$$

and use this fact. Based on the first part we get:

$$-\frac{1}{z^2} = \sum_{n=-\infty}^{-1} ni^{n+1}(z-i)^{n-1} = \sum_{n=-\infty}^{-2} (n+1)i^{n+2}(z-i)^n$$
$$= -\sum_{n=-\infty}^{-2} (n+1)i^n(z-i)^n \quad \forall z \in \{z : 1 < |z-i| < \infty\}.$$

From this follows, that

$$\frac{1}{z^2} = \sum_{n=-\infty}^{-2} (n+1)i^n(z-i)^n,$$

which is the Laurent series in the annulus.

3. Determine the categories of the singularities in (a), (b) and (c) in Exercise 1.1.

Solutions 3. (a) Essential singularity.

(b) Removable singularity.

(c) A simple pole.

4. Find the singularities of the following functions. Determine the types of singularity.

$$f(z) = \frac{\exp(z) - \exp(-z)}{z^4}, \quad f(z) = \exp(z + z^{-1}), \quad f(z) = \frac{z}{\sin^2(z)}.$$

Solutions 4. We notice, that

$$\begin{aligned} f(z) = \frac{\exp(z) - \exp(-z)}{z^4} &= \frac{1}{z^4}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\right) \\ &= \frac{2}{z^3} + \frac{1}{3z} + \frac{2}{5}z + \cdots, \end{aligned}$$

i.e. the only singularity is at $z = 0$, and it is a pole of order 3.

We notice next, that

$$\begin{aligned} f(z) = \exp(z + z^{-1}) &= \sum_{k=0}^{\infty} \frac{(z + z^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{z^k + \binom{k}{1} z^{k-2} + \binom{k}{2} z^{k-4} + \cdots + z^{-k}}{k!}. \end{aligned}$$

We get that $z = 0$ is the only singularity, and that it in fact is an essential singularity.

In the last case we have the possible singularities $z = n\pi$, $n \in \mathbb{Z}$. In the point $z = 0$ we find a simple pole, since

$$\lim_{z \to 0} \frac{z^2}{\sin^2(z)} = 1 \neq 0.$$

In the points $z = n\pi \neq 0$ we find poles of order 2, since

$$\lim_{z \to \pi n} \frac{(z - n\pi)^2 z}{\sin^2(z)} = n\pi \neq 0 \quad \forall n \in \mathbb{Z}\backslash\{0\}$$

5. Determine the category of the singularity in $z_0 = 1$ of the function $f(z) = z\cos\left((z-1)^{-1}\right)$, and calculate $\mathrm{Res}(f, 1)$.

Solutions 5. First we modify the expression $\cos((z-1)^{-1})$:

$$\begin{aligned}\cos\left(\frac{1}{z-1}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n}} \\ &= 1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} - \cdots\end{aligned}$$

From this follows, that

$$\begin{aligned}f(z) = z\cos\left(\frac{1}{1-z}\right) &= (z-1)\cos\left(\frac{1}{1-z}\right) + \cos\left(\frac{1}{1-z}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n-1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n}} \\ &= (z-1) + 1 - \frac{1}{2!(z-1)} - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^3} \\ &\quad + \frac{1}{4!(z-1)^4} - \cdots\end{aligned}$$

The function f has an essential singularity at $z = 1$, and

$$\mathrm{Res}(f, 1) = -\frac{1}{2!} = -\frac{1}{2}.$$