Kompleksianalyysi I kurssin jatkokurssi: Model solutions 1

1. Find the Laurent series of the following functions around the given center.

$$(a)\quad f(z) = ((z+1)^2+1)\sin\frac{1}{z+1} \text{ ja } z_0 = -1$$
$$(b)\quad f(z) = \frac{z-\sin z}{z^3} \text{ ja } z_0 = 0$$
$$(c)\quad f(z) = \frac{z}{(z+1)(z+2)} \text{ ja } z_0 = -2\,.$$

Solutions 1. (a) We remember, that $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{(1+2k)!}$ and put $x = \frac{1}{z+1}$. This gives us

$$\begin{aligned}
f(z) &= \big((z+1)^2+1\big)\sin\left(\frac{1}{z+1}\right)\\
&= \big((z+1)^2+1\big)\left(\frac{1}{z+1} - \frac{1}{3!(z+1)^3} + \frac{1}{5!(z+1)^5} - \frac{1}{7!(z+1)^7} + - \dots\right)\\
&= \frac{(z+1)^2}{z+1} - \frac{(z+1)^2}{3!(z+1)^3} + \frac{(z+1)^2}{5!(z+1)^5} - \frac{(z+1)^2}{7!(z+1)^7} + - \dots\\
&\quad + \frac{1}{z+1} - \frac{1}{3!(z+1)^3} + \frac{1}{5!(z+1)^5} - \frac{1}{7!(z+1)^7} + - \dots\\
&= (z+1) + \frac{5}{6}\frac{1}{z+1} - \frac{19}{120}\frac{1}{(z+1)^3} + \frac{41}{5040}\frac{1}{(z+1)^5} - + \dots
\end{aligned}$$

The series converges $\forall z \in \mathbb{C}\backslash\{-1\}$.

(b) As in part (a), we use the serial expansion of the sine function:

$$\begin{aligned}
f(z) &= \frac{z-\sin(z)}{z^3}\\
&= \frac{1}{z^3}\left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + - \dots\right)\right)\\
&= \frac{1}{z^3}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - + \dots\right)\\
&= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - + \dots
\end{aligned}$$

The series converges $\forall z \in \mathbb{C}$.

(c) We denote $z+2=u$, and get:

$$\begin{aligned}
f(z) &= \frac{z}{(z+1)(z+2)} \\
&= \frac{u-2}{(u-1)u} = \frac{2-u}{u} \cdot \frac{1}{1-u} \\
&= \frac{2-u}{u}\left(1+u+u^2+\dots\right) = (2-u)\left(\frac{1}{u}+1+u+\dots\right) \\
&= \frac{2}{u}+2+2u+\dots-1-u-u^2-\dots \\
&= \frac{2}{u}+1+u+u^2+\dots \\
&= \frac{2}{z+2}+1+(z+2)+(z+2)^2+\dots
\end{aligned}$$

The series converges for all z, such that $0<|z+2|<1$.

2.

Solutions 2. See Exercise Set 2, exercise 1.

3. Let the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ be R and the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ be r, where $rR>1$. Define $a_n = b_{-n}$, when $n<0$. Show, that the series $\sum_{n=-\infty}^{\infty} a_n z^n$ converges uniformly in the compact subsets of the annulus $\{z : 1/r < |z| < R\}$.

Solutions 3. Let $K \subset \{z : \frac{1}{r} < |z| < R\}$ be compact. We choose $r'>0$ and $R'>0$, such that

$$\frac{1}{r} < \frac{1}{r'} < |z| < R' < R \quad \forall z \in K.$$

Next we examine the partial sum

$$S_N(z) = \sum_{|n|<N} a_n z^n, \text{ kun } z \in K.$$

The series converges uniformly if and only if

$$\sup_{z\in K}\left|S_N(z)-\sum_{n=-\infty}^{\infty}a_nz^n\right|\longrightarrow 0 \text{ when } N\to\infty.$$

We get

$$\begin{aligned}\sup_{z\in K}\left|S_N(z)-\sum_{n=-\infty}^{\infty}a_nz^n\right| &= \sup_{z\in K}\left|\sum_{|n|\geq N}a_nz^n\right| \\ &\leq \sup_{z\in K}\sum_{n=N}^{\infty}\big(|a_nz^n|+|a_{-n}z^{-n}|\big) \\ &= \sup_{z\in K}\sum_{n=N}^{\infty}\big(|a_nz^n|+|b_nz^{-n}|\big) \\ &\leq \sum_{n=N}^{\infty}(|a_n||R'|^n+|b_n||r'|^n)\longrightarrow 0 \text{ kun } N\to\infty.\end{aligned}$$

The series converges uniformly.

□

4. Find the Laurent series of the function

$$f(z)=\frac{1}{2-3z+z^2}$$

in the annulus

$$\begin{array}{ll}(a) & \{z:1<|z|<2\}\\ (b) & \{z:\sqrt{2}<|z+i|<\sqrt{5}\}.\end{array}$$

Solutions 4. (a) The case when $z\in\{z:1<|z|<2\}$.
We use partial fraction expansion to get $f(z)$ in the form

$$f(z)=\frac{1}{2-3z+z^2}=\frac{1}{1-z}-\frac{1}{2-z}.$$

We notice, that

$$\frac{1}{1-z}=-\frac{1}{z}\frac{1}{1-\frac{1}{z}},$$

and use the geometric series in the case $|1/z| < 1$, i.e. when $|z| > 1$:

$$\frac{1}{1-z} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right) = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=-\infty}^{-1} z^n.$$

Next we notice that

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-\frac{z}{2}},$$

and using the geometric series in the case $|z/2| < 1$, i.e. when $|z| < 2$, we get:

$$\frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} 2^{-n-1} z^n.$$

It follows, that

$$\begin{aligned}\frac{1}{2-3z+z^2} &= -\sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} 2^{-n-1} z^n \\ &= \ldots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots,\end{aligned}$$

which is the Laurent series in the annulus $\{z : 1 < |z| < 2\}$.

(b) The case when $z \in \{z : \sqrt{2} < |z+i| < \sqrt{5}\}$.

We first use partial fraction expansion in the same way as in part (a), and then notice that

$$\frac{1}{1-z} = \frac{1}{(1+i)-(z+i)} = -\frac{1}{z+i}\frac{1}{1-\left(\dfrac{1+i}{z+i}\right)}.$$

Since $\left|\frac{1+i}{z+i}\right| = \frac{\sqrt{2}}{|z+i|} < 1$, we can again use the geometric series:

$$\frac{1}{1-z} = -\frac{1}{z+i}\frac{1}{1-\left(\frac{1+i}{z+i}\right)} = -\frac{1}{z+i}\sum_{n=0}^{\infty}\left(\frac{1+i}{z+i}\right)^n$$

$$= -\frac{1}{z+i}\sum_{n=-\infty}^{0}\left(\frac{z+i}{1+i}\right)^n$$

$$= -\sum_{n=-\infty}^{0}(1+i)^{-n}(z+i)^{n-1}$$

$$= -\sum_{n=-\infty}^{-1}(1+i)^{-n-1}(z+i)^n.$$

Similarly as in the first part we get

$$\frac{1}{2-z} = \frac{1}{(2+i)-(z+i)} = \frac{1}{2+i}\frac{1}{1-\left(\frac{z+i}{2+i}\right)}.$$

We have $\left|\frac{z+i}{2+i}\right| = \frac{|z+i|}{\sqrt{5}} < 1$, so we can use the geometric series:

$$\frac{1}{2-z} = \frac{1}{2+i}\frac{1}{1-\left(\frac{z+i}{2+i}\right)} = \frac{1}{2+i}\sum_{n=0}^{\infty}\left(\frac{z+i}{2+i}\right)^n$$

$$= \sum_{n=0}^{\infty}(2+i)^{-n-1}(z+i)^n.$$

It follows, that

$$\frac{1}{2-3z+z^2} = -\sum_{n=-\infty}^{-1}(1+i)^{-n-1}(z+i)^n - \sum_{n=0}^{\infty}(2+i)^{-n-1}(z+i)^n$$

$$= \ldots - \frac{1+i}{(z+1)^2} - \frac{1}{z+i} - \frac{1}{2+i} - \frac{(z+i)}{(2+i)^2} - \frac{(z+i)^2}{(2+i)^3} - \ldots,$$

which is the Laurent series in the annulus $\{z : \sqrt{2} < |z+i| < \sqrt{5}\}$.