Descriptive Set Theory

Exercise 6

1. For all $\eta \in pr(R)$ let η_0 and η_1 be s.t. $(\eta,\eta_0),(\eta,\eta_1) \in R$, $\eta_0 \neq \eta_1$ and if $n<\omega$ is largest s.t. $\eta_0 \restriction n = \eta_1 \restriction n$, then $\eta_0(n) < \eta_1(n)$. Then we let $R^* = \{(\eta,\eta_0) \mid \eta \in pr(R)\}$. Clearly R^* uniformizes R and it is Borel: For this it is enough to show that both R^* and $\{(\eta,\eta_1) \mid \eta \in R\}$ are Σ^1_1; this is easy.

2. Repeat the proof of 11.2 except that replace the condition $\sum_{j=0}^{\infty} v(I_{ij}) \leq 2/3$ by $\sum_{j=0}^{\infty} v(I_{ij}) < 1$.

3. By 3.14 we may assume that $rng(f) \subseteq \mathbb{R}$ i.e. that $f: \mathbb{R} \to \mathbb{B}$. For all $\eta \in \omega^n$ let $Y_\eta \subseteq f^{-1}(N_\eta)$ be Borel and s.t. $f^{-1}(N_\eta) \setminus Y_\eta$ is null. Then let $Y_n = \bigcup_{\eta \in \omega^n} Y_\eta$ and finally let $X = \bigcap_{n<\omega} Y_n$. Then X is Borel

$f \restriction X$ is Borel and $\mathbb{R} \setminus X$ is null. So we may choose $g : \mathbb{R} \to \mathbb{B}$ s.t. $g \restriction X = f \restriction X$ and if $r \in \mathbb{R} \setminus X$, then $g(r) = \bar{0}$, where $\bar{0} \in \mathbb{B}$ is s.t. $\bar{0}(n) = 0$ $\forall n < \omega$.

4. Let A be the set of all countable unions U of basic open sets s.t. $U \subseteq V$ and $\mu(U) < 1$. Let ~~FE~~ $\{U_i \mid i < 2^\omega\}$ be an enumeration of A, $\{b_i \mid i < 2^\omega\}$ be an enumeration of $\{(x,y) \in \mathbb{R}^2 \mid y > 2\}$ and $\pi : 2^\omega \to 2^\omega \times 2^\omega$ be a bijection. Let $\rho, \sigma : 2^\omega \to 2^\omega$ be s.t. $\pi(\alpha) = (\rho(\alpha), \sigma(\alpha))$ $\forall \alpha < 2^\omega$.
For all $\alpha < 2^\omega$, let $W_\alpha = \{\sigma(\beta) \mid \beta < \alpha\}$
Then we let $X = \{x_\alpha \mid \alpha < 2^\omega\}$ where $x_\alpha \in V$ are such that for all $\alpha < 2^\omega$

(a) $x_\alpha \notin U_{\rho(\alpha)}$

(b) for all $\beta < \alpha$ and $\gamma \in W_\alpha$, x_α does not belong to the line segment (x_β, b_γ)

(c) for all $\beta < \alpha$, x_β does not belong to $(x_\alpha, b_{\sigma(\alpha)})$

These are easy to find and then X is as wanted.

5. Let $Q = \{q_i \mid i<\omega\}$ and for all $n \in \mathbb{N} \setminus \{0\}$ and $i<\omega$ let I_i^n be an open interval s.t. $\nu(I_i^n) < \frac{1}{n\, 2^{i+2}}$ and $q_i \in I_i^n$.

Then $U^n = \bigcup_{i<\omega} I_i^n$ is open and dense and $\mu(U^n) < \frac{1}{n}$. Let $Z = \bigcap_{n \in \mathbb{N}\setminus\{0\}} U^n$.

Clearly Z is as wanted.