Descriptive set theory

Exercise 5

1. Let $U \subseteq \mathbb{B}$ be open. It is enough to show that both $f^{-1}(U)$ and $f^{-1}(\mathbb{B} \setminus U) = \mathbb{B} \setminus f^{-1}(U)$ are Σ^1_1. This is clear since e.g.
$f^{-1}(U) = pr(graph(f) \cap (\mathbb{B} \times U))$

2. For all $\eta \in \omega^n$, $n < \omega$, let $Y_\eta = \bigcap_{m \leq n} X_{\eta \upharpoonright m}$

Then it is easy to see that
$\mathcal{A}\{Y_\eta \mid \eta \in \omega^{<\omega}\} = \mathcal{A}\{X_\eta \mid \eta \in \omega^{<\omega}\}$.
Thus we may assume that for $\eta, \xi \in \omega^{<\omega}$, if $\eta \subseteq \xi$ then $X_\eta \subseteq X_\xi$.
But then
$\mathcal{A}\{X_\eta \mid \eta \in \omega^{<\omega}\} = \bigcap_{n < \omega} X_n$ where
$X_n = \bigcup_{\eta \in \omega^n} X_\eta$.

3. Easy induction on α.

4. $T_\alpha \leq X^*_\eta \Leftrightarrow rk(T_\alpha) \leq rk(X^*_\eta)$
$\Leftrightarrow rk(X^*_\eta) \geq \alpha$.

5. By an easy induction one can show the following: Let $\eta : n \to T$ s.t. $n > 0$ and $\eta \in \sigma(T)$. Then $rk(\eta; \sigma(T)) = rk(\eta(n-1); T)$. The claim follows from this immediately.

6. $T' \not\leq T \iff rk(T') > rk(T) \iff$

$rk(T') \geq rk(\sigma(T))$

$\iff \sigma(T) \leq T'$