Descriptive Set theory

Exercise 4

1. $\bigcup_{i<\omega} pr(C_i) = pr(\bigcup_{i<\omega} C_i)$

For intersections let $C = \{(\eta, (\xi_i)_{i<\omega}) \in \mathbb{B} \times \mathbb{B}^\omega \mid \forall i<\omega\ ((\eta, \xi_i) \in C_i)\}$. If $(\eta, (\xi_i)_{i<\omega}) \notin C$

Then there is $k<\omega$ s.t. $(\eta, \xi_k) \notin C_k$

and thus there is $n<\omega$ s.t. $N_{\eta\restriction n, \xi_k\restriction n} \cap C_k = \emptyset$.

But then $U = \{(\eta', (\xi'_j)_{j<\omega}) \in \mathbb{B} \times \mathbb{B}^\omega \mid \eta\restriction n \subseteq \eta'$ and $\xi_k\restriction n \subseteq \xi'_k\}$ is open

and $U \cap C = \emptyset$. Thus C is closed

and clearly $\bigcap_{i<\omega} pr(C_i) = pr(C)$.

2. If $f: \mathbb{B} \to \mathbb{B}$ is Borel then also

$h: \mathbb{B} \times \mathbb{B} \to \mathbb{B} \times \mathbb{B}$, $h(\eta, \xi) = (f(\eta), \xi)$

is Borel and then the claim follows

by an easy induction once one notices

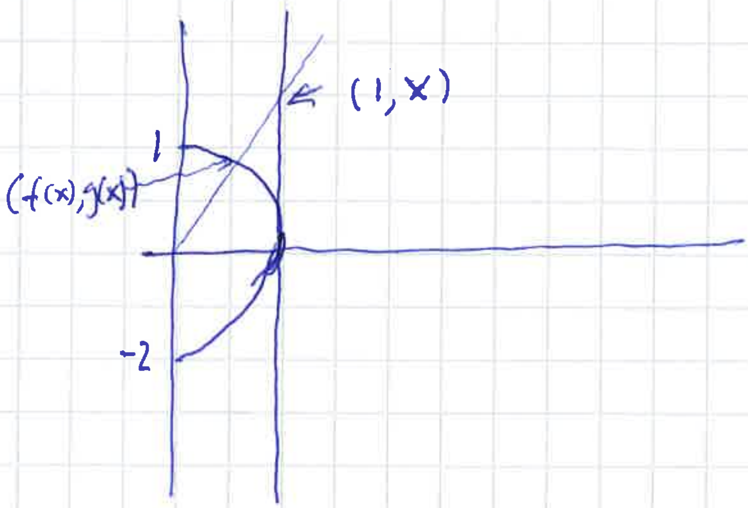
that $pr(h^{-1}(X)) = f^{-1}(pr(X))$ and

$\mathbb{B}^2 \setminus h^{-1}(X) = h^{-1}(\mathbb{B}^2 \setminus X)$.

3. $f(X) = \bigcup_{0<n\in\omega} f(X \cap ([-n,n] \times [-n,n]))$ and since $X \cap ([-n,n]\times[-n,n])$ is compact also $f(X \cap ([-n,n]\times[-n,n]))$ is compact and thus also closed.

4. $x \mapsto (x,0,0)$ is clearly a homeomorphism.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be as in the picture below



Then we define $H^{-1}(y,x) = (y, f(x), g(x))$

Clearly H is as wanted.

6. Let $X \subseteq \mathbb{R}^2$ be Borel s.t. $pr(X)$ is not Borel. Let $Z = H^{-1}(X) \subseteq \mathbb{R}^3$.

Then Z is Borel. Let $Y = \bigcup_{z \in Z} \bar{B}(z,1)$.

Let $C = A \cap Y$. Now

$(x,0,0) \in C \iff x \in pr(X)$. Thus

C is not Borel and since A is Borel, Y can not be Borel.