Descriptive Set Theory

Exercise 3

1. Only (b) $\Rightarrow$ (c) needs a proof. ((c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b) are clear) Prove this by induction on X being Borel i.e. use 1.5.3

2. It is enough to show that $g = f \restriction (I_r)^m$ is continuos in a co-meager set $X \subseteq (I_r)^m$

Let ~~d~~ and h be as ~~..~~ X and f are in 3.14. By 3.13 there is co-meager $X \subseteq (I_r)^m$ s.t. $h^{-1} \circ g$ is continuous in X. But then $f \restriction X = (h \circ (h^{-1} \circ g)) \restriction X$ is continuous (h is continuous)

3. Let (T, π) be a Borel**-code for X. We need to find Borel*-code for X. For $m < n$ and $\eta \in \omega^n$, let $\eta^m \in \omega^{n-m}$ be such that for all $i < n-m$, $\eta^m(i) = \eta(m+i)$.

For all $\eta \in L(T)$, let (T_η, π_η) be a Borel*-code for $\pi(\eta)$.

Let $T^* = T \cup \{\eta \in \omega^n \mid n < \omega, \exists m < n \; (\eta \restriction m \in L(T) \text{ and } \eta^m \in \pi_{\eta \restriction m}(\eta \restriction m))\}$ and π^* ~~...~~ such that

For all $\eta \in L(T^*)$ if m is such that $\eta \restriction m \in L(T)$, then $\pi^*(\eta) = \pi_{\eta \restriction m}(\eta^m)$

Clearly (T^*, π^*) is a Borel*-code for X.

4. Essentially as the proof of 4.8

5. It is enough to prove by induction on $n < \omega$, that for all $m < \omega$, if $X \subseteq \mathbb{R}^n \times \mathbb{B}^m$ is Borel and uncountable, then X contains a perfect set.
The case $n=0$ follows from 4.8.

So suppose $n = k+1$: If for some $q \in \mathbb{Q}$ $\{ \cancel{\ldots}\ x = (x_0, .., x_{n+m-1}) \in \mathbb{R}^n \times \mathbb{B}^m \mid x \in X$ and $x_k = q \}$ is uncountable, the claim follow from the induction assumption for $\mathbb{R}^k \times \mathbb{B}^m$. So we may assume that $\{ x \in X \mid x_k \notin \mathbb{Q} \}$ is uncountable. But then the claim follows from the induction assumption for $\mathbb{R}^k \times \mathbb{B}^{m+1}$.

6, (i) $\Rightarrow$ (iii): Now Σ_α is closed under complements and countable unions.

(iii) $\Rightarrow$ (ii): $\Sigma_\alpha \subseteq \Sigma_{\alpha+1} \subseteq \text{Borel} = \Sigma_\alpha$

(ii) $\Rightarrow$ (i): $\Pi_\alpha \subseteq \Sigma_{\alpha+1} = \Sigma_\alpha$. By looking complements, on can see that also $\Sigma_\alpha \subseteq \Pi_\alpha$