Descriptive set theory

Exercise 1.

1. a) apply pairing twice

   b) By foundation $a \neq \{a,b\}$ and $c \neq \{c,d\}$. Thus by extensionality either (1) $a = c$ and $\{a,b\} = \{c,d\}$ or (2) $a = \{c,d\}$ and $c = \{a,b\}$. From (2) it follows that $a \in c \in a$ which contradict foundation. Thus (1) holds. But then by extensionality and (1) $b = d$.

2. By adding one point to a lin. order as a largest elem., one gets a lin. order. Thus $\alpha+1$ is linearly ordered by $\in$. It is easy to see that $\alpha+1$ is transitive and thus $\alpha+1$ is an ordinal. If $\alpha \leq \beta \leq \alpha+1$, then by 1.3.3 $\alpha \subseteq \beta \subseteq \alpha+1$ and thus $\beta = \alpha$ or $\beta = \alpha+1$.

3. If $\beta \in \alpha$, then $\beta \subseteq \alpha$ (by transitivity of α) and a subset of a lin. order is a lin. order and so β is linearly ordered by $\in$. To show that β is transitive, suppose $\delta \in \gamma \in \beta$. Then $\delta, \gamma \in \alpha$ and thus $\delta \in \beta$ or $\delta = \beta$ or $\beta \in \delta$. The last two possibilities contradict foundation.

4. For all $\alpha, \beta \in a$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ and thus it is easy to see that $\cup a$ is an ordinal. If $\alpha \leq \gamma$ for all $\alpha \in a$, $\alpha \subseteq \gamma$ and thus $\cup a \subseteq \gamma$ and so $\cup a \leq \gamma$.

5. If such f exists, by replacement $\{f(n) \mid n \in \omega\}$ is a set and this contradicts foundations

6. By 4. $\cup a$ is an ordinal.

If $\kappa = \cup a$ is not a cardinal, there is $\alpha \in \kappa$ and $1-1$ $f : \kappa \to \alpha$. But then there is $\lambda \in a$ s.t. $\alpha \in \lambda$ and $f \restriction \lambda$ is $1-1$ from λ to α $\lightning$.