

20. Surfaces of constant Gaussian curvature

S = an open subset of any plane \Rightarrow Gaussian curvature $K=0$.

S = an open subset of a sphere of radius $R \Rightarrow K = \frac{1}{R^2}$

There also are surfaces with constant negative Gaussian curvature. Later we may prove that a compact surface always has a point at which $K > 0$. Thus a surface with constant ^{negat.} Gaussian curvature cannot be bounded.

Exercise: Let $\alpha(u) = (g(u), h(u), 0)$ be a regular curve, $h > 0$. Rotate α about the x -axis \Rightarrow a smooth surface S , parametrized by.

$$X(u, v) = (g(u), h(u)\cos v, h(u)\sin v).$$

Then: $X_u = (g', h'\cos v, h'\sin v)$

$$X_v = (0, -h\sin v, h\cos v)$$

$$X_u \times X_v = (hh', -g'h\cos v, -g'h\sin v)$$

Unit normal vector $N_x = \frac{(h', -g'\cos v, -g'\sin v)}{[(h')^2 + (g')^2]^{1/2}}$

$$X_{uu} = (g'', h''\cos v, h''\sin v)$$

$$X_{uv} = (0, -h'\sin v, h'\cos v)$$

$$X_{vv} = (0, -h\cos v, -h\sin v)$$

Find E, F, G, L, N, M .

Then $K = \frac{LN - M^2}{EG - F^2} = \frac{g'(g''h' - h''g')}{h((h')^2 + (g')^2)^2}$

Example Let $\alpha: (0, \infty) \rightarrow \mathbb{R}^3$, $u \mapsto (u - \tanh u, \frac{1}{\cosh u}, 0)$.
 Rotate α about the x -axis \Rightarrow surface S , parametrized by

$$X(u, v) = \left(\underbrace{u - \tanh u}_{g(u)}, \underbrace{\frac{1}{\cosh u}}_{h(u)} \cos v, \frac{1}{\cosh u} \sin v \right)$$

Previous exercise \Rightarrow the Gaussian curvature K of S is

$$\frac{g'(g''h' - h''g')}{h((h')^2 + (g')^2)^{3/2}}$$

Calculate K :

$$g(u) = u - \frac{\sinh u}{\cosh u} \Rightarrow g'(u) = 1 - \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u} = 1 - \frac{1}{\cosh^2 u}$$

$$= \frac{\cosh^2 u - 1}{\cosh^2 u} = \frac{\sinh^2 u}{\cosh^2 u}$$

$$\Rightarrow g'' = - \left(\frac{0 - 2 \cosh u \sinh u}{\cosh^4 u} \right) = \frac{2 \sinh u}{\cosh^3 u}$$

$$h(u) = \frac{1}{\cosh u} \Rightarrow h'(u) = \frac{-\sinh u}{\cosh^2 u}$$

$$\Rightarrow h'' = \frac{-\cosh^3 u + \sinh u \cdot 2 \sinh u \cosh u}{\cosh^4 u} = \frac{-1}{\cosh u} + \frac{2 \sinh^2 u}{\cosh^3 u}$$

Then

$$g''h' - h''g' = - \frac{2 \sinh u}{\cosh^3 u} \cdot \frac{\sinh u}{\cosh^2 u} - \frac{\sinh^2 u}{\cosh^2 u} \left(\frac{-1}{\cosh u} + \frac{2 \sinh^2 u}{\cosh^3 u} \right)$$

$$= \frac{\sinh^2 u}{\cosh^3 u} - \frac{2 \sinh^2 u + 2 \sinh^4 u}{\cosh^5 u}$$

$$= \frac{\sinh^2 u}{\cosh^3 u} - \frac{2 \sinh^2 u (1 + \sinh^2 u)}{\cosh^5 u}$$

$$= \frac{\sinh^2 u - 2 \sinh^2 u}{\cosh^3 u} = \frac{-\sinh^2 u}{\cosh^3 u}$$

and

$$(h')^2 + (g')^2 = \frac{\sinh^2 u}{\cosh^4 u} + \frac{\sinh^4 u}{\cosh^4 u} = \frac{\sinh^2 u (1 + \sinh^2 u)}{\cosh^4 u} = \frac{\sinh^2 u}{\cosh^2 u}$$

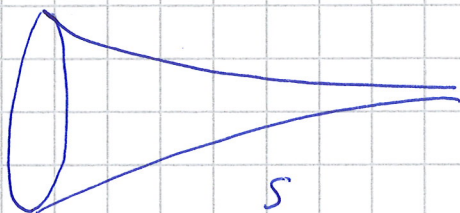
Then

$$K = \frac{g'(g''h' - h''g')}{h((h')^2 + (g')^2)^2}$$

$$= \frac{\frac{\sinh^2 u}{\cosh^2 u}}{g'} \cdot \frac{\cosh u}{1/h} \cdot \left(-\frac{\sinh^2 u}{\cosh^3 u} \right) \cdot \left(\frac{\cosh^2 u}{\sinh^2 u} \right)^2$$

$$= -\frac{\cosh^5 u}{\cosh^5 u} \cdot \frac{\sinh^4 u}{\sinh^4 u} = \underline{\underline{-1}}$$

The surface S has constant negative Gaussian curvature $K = -1$. The surface S is called a pseudosphere.



Lemma 20.1. Let X be a compact subset of \mathbb{R}^3 , and let $d: X \rightarrow \mathbb{R}$ be a continuous function. Then there are points $p, q \in X$ s.t.

$$d(q) \leq d(x) \leq d(p),$$

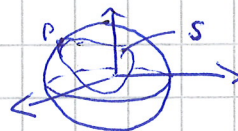
for all $x \in X$. (Then d obtains its maximum value at p and its minimum value at q .)

Proof. \square .

Proposition 20.2. Let S be a smooth compact surface in \mathbb{R}^3 . Then there is a point $p \in S$ such that the Gaussian curvature $K(p)$ of S at p is > 0 .

proof. Let $d: \mathbb{R}^3 \rightarrow \mathbb{R}, v \mapsto \|v\|^2$. Then d is continuous.
 S compact $\Rightarrow \exists p \in S : d(x) \leq d(p) \forall x \in S$. Let

$$B = B_{\|p\|}(0) = \{x \in \mathbb{R}^3 \mid \|x\| \leq \|p\|\}.$$



$\forall x \in S$, then $\|x\|^2 = d(x) \leq d(p) = \|p\|^2 \Rightarrow \|x\| \leq \|p\|$. Then $S \subseteq B$.

Let $\gamma: (a, b) \rightarrow S$ be a unit-speed curve. Assume $0 \in (a, b)$ and $\gamma(0) = p$. Then $d \circ \gamma$ has a local maximum at $t=0$.
 Then

$$\left. \frac{d}{dt} d(\gamma(t)) \right|_{t=0} = 0$$

$$\left. \frac{d^2}{dt^2} d(\gamma(t)) \right|_{t=0} \leq 0.$$

Here $(d \circ \gamma)(t) = \gamma(t) \cdot \gamma(t)$

$$\Rightarrow \left. \frac{d}{dt} (d \circ \gamma)(t) \right|_{t=0} = 2\gamma'(0) \cdot \gamma(0)$$

$$\Rightarrow 0 = \left. \frac{d}{dt} (d \circ \gamma)(t) \right|_{t=0} = 2\gamma'(0) \cdot \gamma(0) = 2\gamma'(0) \cdot p.$$

Write $\gamma'(0) = u$. Then $u \in T_p S$. Since γ was arbitrary, it follows that $u \cdot p = 0 \forall u \in T_p S$. Thus p is a normal vector for $T_p S$. Moreover,

$$\left. \frac{d^2}{dt^2} (d \circ \gamma) \right|_{t=0} = \left. \frac{d}{dt} (2\gamma' \cdot \gamma) \right|_{t=0} = 2\gamma' \cdot \gamma' + 2\gamma'' \cdot \gamma.$$

$$\Rightarrow 0 \geq \left. \frac{d^2}{dt^2} (d \circ \gamma) \right|_{t=0} = 2\gamma'(0) \cdot \gamma''(0) + 2\gamma''(0) \cdot \gamma(0)$$

$$= 2\underbrace{u \cdot u}_=1 + 2p \cdot \gamma''(0).$$

since γ is unit-speed

$$\Rightarrow -2 \geq 2p \cdot \gamma''(0)$$

$$\Rightarrow -1 \geq p \cdot \gamma''(0).$$

The vector $\frac{p}{\|p\|}$ is a unit normal vector at p . Then the normal curvature of γ at p is

$$k_n = \gamma''(0) \cdot \frac{p}{\|p\|} \leq \frac{-1}{\|p\|}$$

(or $k_n \geq \frac{1}{\|p\|}$ if we have $N = \frac{-p}{\|p\|}$).

Since this holds for any p , it follows that for the principal curvatures k_1 and k_2 of S at p , $k_1, k_2 \leq \frac{-1}{\|p\|}$.
(or $k_1, k_2 \geq \frac{1}{\|p\|}$). Thus, at p , (cor. 17.5)

$$K \geq \frac{1}{\|p\|^2} > 0. \quad \square$$

Corollary 20.3. No compact surface in \mathbb{R}^3 can have negative curvature. \square

Definition 20.4. A smooth surface in \mathbb{R}^3 is called minimal, if its mean curvature H is zero everywhere.

$$S \text{ minimal} : H = \frac{1}{2}(k_1 + k_2) = 0$$

$$\Rightarrow k_1 = -k_2$$

$$\Rightarrow K = k_1 k_2 = -k_2^2 \leq 0.$$

Corollary 20.5. No minimal surface in \mathbb{R}^3 can be compact. \square