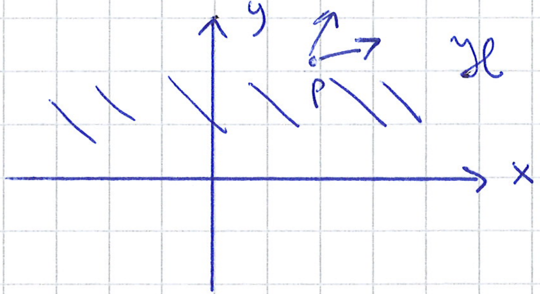
25. Hyperbolic Geometry

The upper half-plane is

$$\mathcal{H} = \{(v,w) \in \mathbb{R}^2 \mid w > 0\} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$



Let $p = (u,v) \in \mathcal{H}$. Then $v > 0$. Let $w = (w_1, w_2) (= w_1 + i w_2)$ and $w' = (w_1', w_2')$ be vectors at p. Define the inner product of w and w' at p to be

$$\langle w, w' \rangle_p = \frac{1}{v^2} \underbrace{(w_1 w_1' + w_2 w_2')}_{\text{Euclidean inner product of } w \text{ and } w'.}$$

Define the angle θ between the vectors w and w' with the equation

$$\cos\theta = \frac{\langle w, w' \rangle_p}{\|w\|_p \, \|w'\|_p} \qquad (*)$$

as in the Euclidean case. The terms $\frac{1}{v^2}$, where $p = (u,v)$ cancel out in $(*)$. Therefore, we obtain

Proposition 25.1. Hyperbolic angles in $\mathcal{H}$ are the same as Euclidean angles. $\square$

The hyperbolic plane $\mathcal{H}$ can be parametrized just like the euclidean plane: $X(u,v) = (u,v)$. Then

$$X_u = (1,0) \quad \text{and} \quad X_v = (0,1).$$

As in the euclidean case, define the coefficients E, F and G of the first fundamental form by using inner products. Then

$$E(u,v) = (X_u \cdot X_u)(u,v) = \frac{1}{v^2}(1,0)\cdot(1,0) = \frac{1}{v^2}$$

$$F(u,v) = (X_u \cdot X_v)(u,v) = 0$$

$$G(u,v) = (X_v \cdot X_v)(u,v) = \frac{1}{v^2}(0,1)\cdot(0,1) = \frac{1}{v^2}$$

Thus the first fundamental form of $\mathcal{H}$ is

$$E(du)^2 + 2F\,du\,dv + G(dv)^2 = \frac{1}{v^2}\left((du)^2 + (dv)^2\right).$$

$F=0$: Gauss Theorem $\Rightarrow$

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right).$$

Here $\quad G = \frac{1}{v^2} \Rightarrow G_u = 0,$

and $\quad E = \frac{1}{v^2} = v^{-2} \Rightarrow E_v = -2v^{-3} = \frac{-2}{v^3},$

$\quad EG = \frac{1}{v^4} \Rightarrow \sqrt{EG} = \frac{1}{v^2}.$

Then

$$K = -\frac{1}{2}\cdot v^2\left(\frac{\partial}{\partial v}\left(v^2 \cdot \frac{-2}{v^3}\right)\right) = v^2 \frac{\partial}{\partial v}(v^{-1})$$

$$= v^2\left(-\frac{1}{v^2}\right) = -1.$$

Thus the hyperbolic plane has constant Gaussian curvature $K = -1$.

Theorem that was mentioned earlier but not proved, says the following:

Let $\gamma(s)$ be a unit-speed simple closed curve on a surface patch X of length $\ell(\gamma)$. Assume γ is positively oriented. Then,

$$\int_0^{\ell(\gamma)} k_g \, ds = 2\pi - \int_{\mathrm{int}(\gamma)} K \, dA_X ,$$

where k_g is the geodesic curvature of γ, K is the Gaussian curvature of X and dA_X is the area element.

Assume $\gamma \subset \mathcal{H}$ is a geodesic, and assume γ is a unit-speed simple closed curve. By the result above

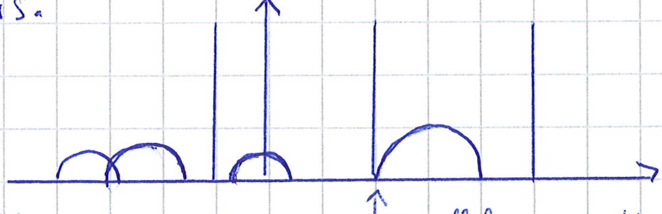
$$\int_0^{\ell(\gamma)} \underbrace{k_g}_{=0} \, ds = 2\pi - \int_{\mathrm{int}(\gamma)} - \, dA_X .$$

$$\Rightarrow \underbrace{\int_{\mathrm{int}(\gamma)} dA_X}_{>0} = -2\pi$$

This is impossible. $\Rightarrow$ no simple closed curve can be a geodesic in $\mathcal{H}$.

In fact we have:

<u>Theorem 25.2.</u> The geodesics in $\mathcal{H}$ are the half-lines parallel to the imaginary axis and the semicircles with centres on the real axis.



proof: Skip.

parallel, since these do not intersect in $\mathcal{H}$

The geodesics in $\mathcal{H}$ are called <u>hyperbolic lines</u>. Two hyperbolic lines are called <u>parallel</u>, if they do not intersect.

There are some properties of hyperbolic lines:

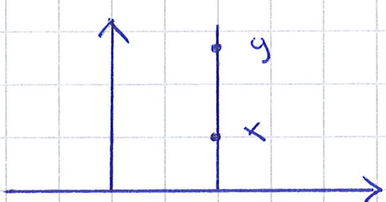
<u>Proposition 25.3.</u>

1) Let $x, y \in \mathcal{H}$, $x \neq y$. Then there is a unique hyperbolic line through x and y.

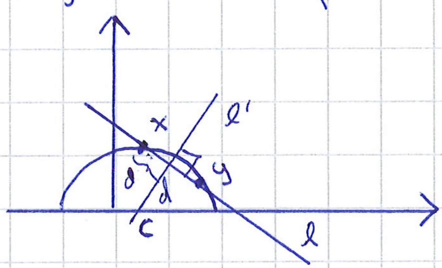
2) The parallel axiom does not hold in $\mathcal{H}$.

<u>proof.</u>

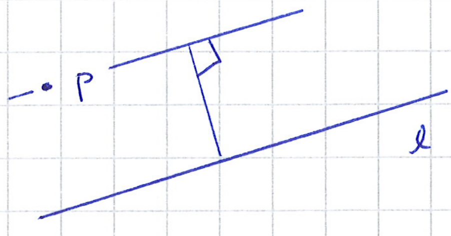
1) Assume first the unique euclidean line passing through x and y is parallel to the imaginary axis. Then the unique hyperbolic line passing through x and y is the half-line containing them.



Assume then that the unique euclidean line ℓ passing through x and y is not parallel to the imaginary axis.
Let ℓ' be the perpendicular bisector of the line segment $[x, y]$.
Then ℓ' intersects the real axis at some point c.
Thus the unique hyperbolic line passing through x and y is the semicircle with center c and radius $\|x - c\| = \|y - c\|$.

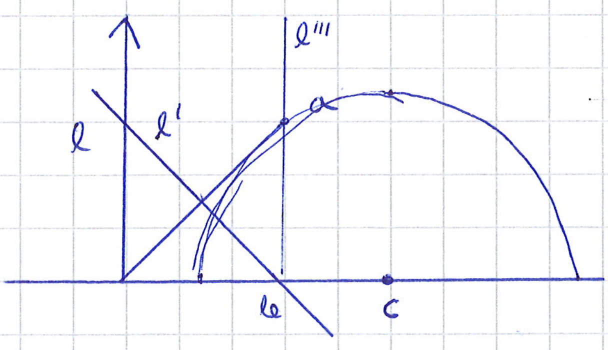


2) <u>Parallel axiom:</u> Let p be a point in a plane. Assume p is not on a straight line l. Then there is a unique straight line in the plane that passes through p and is parallel to l.



Parallel axiom does not hold in $\mathcal{H}$:

Let l be the imaginary axis. Let $a \in \mathcal{H}$, $a \notin l$. We may assume $\mathrm{Re}(a) > 0$. Let l' be the perpendicular bisector of the line joining a and the origin. Then l' intersects the real axis at some point $b > 0$. Let c be a point on the real axis $c > b$.



points on this side of l' are closer to a than to the origin

Let l'' be the semicircle in $\mathcal{H}$ with centre c and passing through a. Then $0 \notin l''$. Let l''' be the half-line through a parallel to the imaginary axis. Then $a \in l''$, $a \in l'''$ and both l'' and l''' are parallel to the imaginary axis. □

Hyperbolic distance

Let $a, b \in \mathcal{H}$. We define the hyperbolic distance $d_{\mathcal{H}}(a,b)$ to be the length of the hyperbolic line segment joining a and b. This distance is the hyperbolic length of the shortest curve joining a and b.

Proposition 25.4. Let $a, b \in \mathcal{H}$. Then the hyperbolic distance between a and b is

$$d_{\mathcal{H}}(a,b) = 2\tanh^{-1}\left(\frac{\|b-a\|}{\|b-\bar{a}\|}\right),$$

where $\bar{a}$ is the complex conjugate of a.

proof. There are two cases, depending on whether the hyperbolic line joining a and b is a semi-circle or a half-line. We will prove the case of a semi-circle.

Assume a and b lie on a semicircle with center $c \in \mathbb{R}$ and radius r. Parametrize the semicircle by

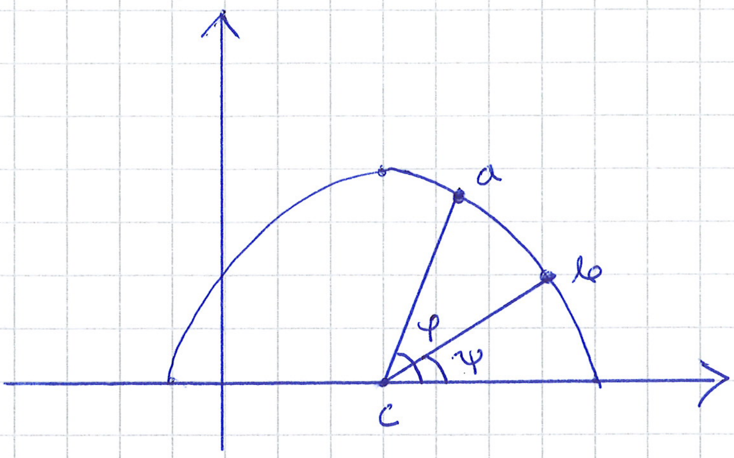
$$u = c + r\cos\theta$$
$$v = r\sin\theta .$$

Write $d_{\mathcal{H}}(a,b) = d$, and denote $\frac{d}{d\theta}$ by $'$. Then

$$d = \int_{\varphi}^{\psi} \sqrt{\frac{(u')^2 + (v')^2}{v^2}}\, d\theta$$

$$= \int_{\varphi}^{\psi} \sqrt{\frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2\sin^2\theta}}\, d\theta = \int_{\varphi}^{\psi} \frac{1}{\sin\theta}\, d\theta ,$$

where $\varphi = \arg(a-c)$, $\psi = \arg(b-c)$. (Notice that d does not depend on the radius of the semicircle.)



Let's do $\int \frac{1}{\sin x}\,dx$:

Let $x = 2y \Rightarrow dx = 2dy$.

$$\int \frac{1}{\sin x}\,dx = \int \frac{2}{\sin 2y}\,dy = \int \frac{1}{\sin y \cos y}\,dy$$

$$= \int \frac{1}{\tan y \cos^2 y}\,dy = \int \frac{1}{\tan y}(\tan y)'\,dy$$

$$= \ln(\tan y) + C = \ln(\tan \tfrac{x}{2}) + C$$

Thus

$$d = \int_{\varphi}^{\psi} \frac{1}{\sin\theta}\,d\theta = \ln(\tan \tfrac{\theta}{2})\Big|_{\varphi}^{\psi}$$

$$= \ln(\tan \tfrac{\psi}{2}) - \ln(\tan \tfrac{\varphi}{2})$$

$$= \ln\left(\frac{\tan \frac{\psi}{2}}{\tan \varphi/2}\right).$$

Then

$$\tanh \frac{d}{2} = \frac{\sinh \frac{d}{2}}{\cosh \frac{d}{2}} = \frac{e^{\frac{d}{2}} - e^{-\frac{d}{2}}}{e^{\frac{d}{2}} + e^{-\frac{d}{2}}} = \frac{e^d - 1}{e^d + 1},$$

but

$$e^d = e^{\ln(\,)} = \frac{\tan \psi/2}{\tan \varphi/2}$$

$$\Rightarrow \quad \frac{e^d - 1}{e^d + 1} = \frac{\tan \psi/2 - \tan \varphi/2}{\tan \psi/2 + \tan \varphi/2}$$

Thus

$$(\cos\tfrac{\psi}{2}\cos\tfrac{\varphi}{2}$$

$$\tanh\frac{d}{2} = \frac{\tan\psi/2 - \tan\varphi/2}{\tan\psi/2 + \tan\varphi/2}$$

$$= \frac{\sin\psi/2\cos\varphi/2 - \sin\varphi/2\cos\psi/2}{\sin\psi/2\cos\varphi/2 + \sin\varphi/2\cos\psi/2}$$

$$= \frac{\sin\frac{\psi-\varphi}{2}}{\sin\frac{\psi+\varphi}{2}}. \qquad (*)$$

On the other hand,

$$a = c + (r\cos\varphi, r\sin\varphi) = (c + r\cos\varphi, r\sin\varphi)$$
$$b = (c + r\cos\psi, r\sin\psi).$$

Thus

$$\|b-a\|^2 = r^2(\cos\psi - \cos\varphi)^2 + r^2(\sin\psi - \sin\varphi)^2$$

$$= r^2\big(\cos^2\psi + \cos^2\varphi - 2\cos\psi\cos\varphi + \sin^2\psi + \sin^2\varphi - 2\sin\psi\sin\varphi\big)$$

$$= 2r^2(1 - \cos(\psi-\varphi)), \quad 2\cos^2 x - 1 = \cos 2x$$

$$= 2r^2\left[1 - \left(2\cos^2\left(\tfrac{\psi-\varphi}{2}\right) - 1\right)\right]$$

$$= 4r^2\left[1 - \cos^2\left(\tfrac{\psi-\varphi}{2}\right)\right]$$

$$= 4r^2\sin^2\frac{\psi-\varphi}{2}. \qquad (1)$$

Let $\bar{a}$ = the complex conjugate of a. Similarly one sees that

$$\|b-\bar{a}\|^2 = 4r^2\sin^2\frac{\psi+\varphi}{2}. \qquad (2)$$

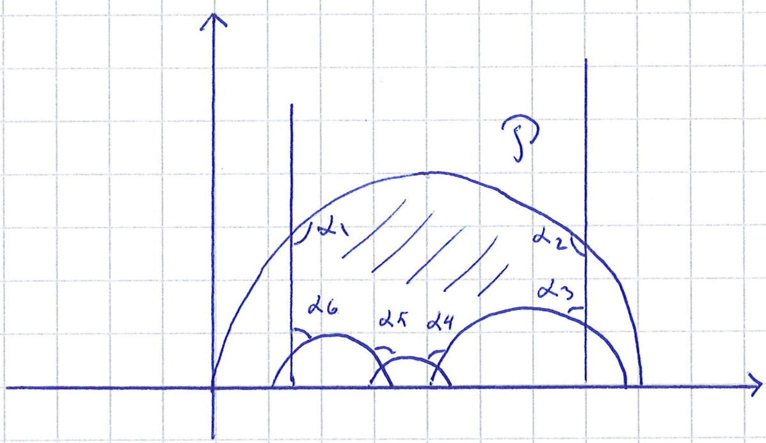
Equations $*$, 1 and 2 $\Rightarrow$

$$\tanh \frac{d}{2} = \frac{\| b - a \|}{\| b - \bar{a} \|} . \qquad \square$$

Definition: A hyperbolic polygon is a polygon $\in \mathcal{H}$, whose sides are segments of hyperbolic lines.

Theorem 25.5. Let P be a hyperbolic n-gon in $\mathcal{H}$ with internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. The hyperbolic area of P is

$$A(P) = (n-2)\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_n .$$



Remark: If P is a hyperbolic triangle, then Theorem 25.5 implies that

$$A(P) = \pi - \alpha - \beta - \gamma$$

where α, β and γ are the angles of P.

Then $A(P) > 0 \Rightarrow \alpha + \beta + \gamma < \pi$.

(For euclidean triangle, $\alpha + \beta + \gamma = \pi$.)

Proof of Theorem 25.5:

Let $a_1, \dots, a_n$ be the vertices of P.
Let C be the boundary of P.
Then C consists of line segments, denoted by $a_1a_2, a_2a_3, \dots, a_na_1$.
$\forall\, i \in \{1, \dots, n\}$, let α_i be the internal angle at a_i.

The 1^{st} fundamental form: $\frac{1}{v^2}\left((du)^2+(dv)^2\right)$.

Recall: The area of a region $R \subset X(U)$, where $X: U \to S$ is a regular patch of a smooth surface, is

$$A(R) = \int_{R'} \|X_u \times X_v\|\, du\, dv$$

$$= \int_{R'} \sqrt{EG-F^2}\, du\, dv,$$

where $R = X(R')$.

For $\mathcal{H}$: $EG - F^2 = \frac{1}{v^2}\cdot\frac{1}{v^2} - 0 = \frac{1}{v^4}$

$$\Rightarrow \sqrt{EG-F^2} = \frac{1}{v^2}.$$

$\Rightarrow$ The area of P is

$$A(P) = \int_P \frac{1}{v^2}\, du\, dv.$$

We evaluate this integral by using Green's Theorem:

$$\int_C p\, du + q\, dv = \int_P \left(\frac{\partial q}{\partial u} - \frac{\partial p}{\partial v}\right) du\, dv,$$

where $p(u,v)$ and $q(u,v)$ are smooth functions.

Choose $p = \frac{1}{v}$, $q = 0$.

Then $\frac{\partial p}{\partial v} = \frac{-1}{v^2}$, $\frac{\partial q}{\partial u} = 0$.

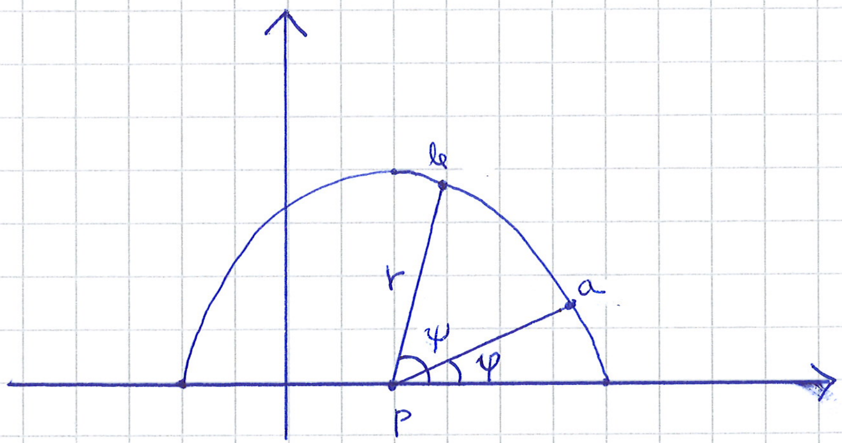
Green's Theorem $\Rightarrow \int_{D} \frac{1}{v^2} \, du \, dv = \int_{C} \frac{1}{v} \, du$.

To evaluate the integral above, we need the following lemma:

Lemma 25.6:

Let l be a hyperbolic line in $\mathcal{H}$, assume l is a semicircle with center $p \in \mathbb{R}$. Let a and b be endpoints of a segment of l. Assume the radius vectors joining p to a and p to b make angles φ and ψ, respectively, with the positive real axis.
Then
$$\int_{l_{ab}} \frac{du}{v} = \varphi - \psi,$$

where l_{ab} is the segment of l with endpoints a and b.



Proof of Lemma 25.6:

Let r be the radius of the semicircle.
Parametrize l by

$$u = r\cos\theta + p$$
$$v = r\sin\theta$$

Then

$$\int_{l_{ab}} \frac{du}{v} = \int_{\varphi}^{\psi} \frac{1}{r\sin\theta}(-r\sin\theta)\,d\theta$$

$$= -\int_{\varphi}^{\psi} d\theta = \varphi - \psi. \quad \square$$

Return to the proof of Theorem 25.5:

vertices: $a_1, \dots, a_n$

a_{n+1} means a_1

sides with endpoints a_i and a_{i+1},
angles, as in the Lemma, φ_i and ψ_i.

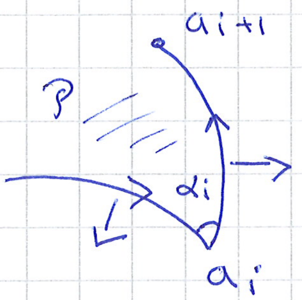
Then

$$A(P) = \int_P \frac{1}{v^2}\,du\,dv \overset{\text{Green}}{=} \sum_{i=1}^{n} \int_{l_{a_i,a_{i+1}}} \frac{1}{v}\,du$$

$$= \sum_{i=1}^{n} (\varphi_i - \psi_i) \qquad (\text{Lemma } 25.4)$$

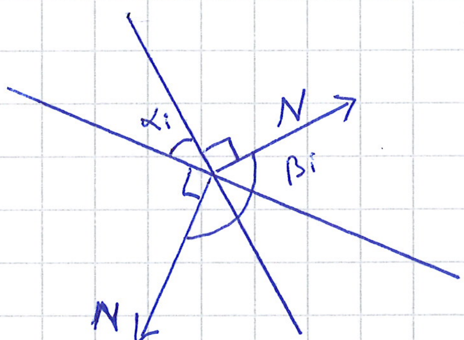
Simplify this sum:

Traverse the boundary of P counterclockwise. Consider the change of the outward-pointing normal of P:



Traverse the side with endpoints a_i and a_{i+1}: The outward normal rotates through an angle $\psi_i - \varphi_i$.

At the vertex a_i:



The change is

$$\beta_i = 2\pi - \left(\frac{\pi}{2} + \alpha_i + \frac{\pi}{2}\right)$$

$$= \pi - \alpha_i.$$

Traverse the boundary of P:

The outward normal rotates through an angle

$$\sum_{i=1}^{n} \left[\pi - \alpha_i + (\psi_i - \varphi_i)\right] = n\pi + \sum_{i=1}^{n} (\psi_i - \varphi_i - \alpha_i).$$

However, the angle of rotation must be 2π.

Then

$$2\pi = n\pi + \sum_{i=1}^{n} (\psi_i - \varphi_i - \alpha_i)$$

$$\Rightarrow \quad \mathcal{A}(P) = \int_P \frac{1}{v}\, du\, dv = \sum_{i=1}^{n} (\varphi_i - \psi_i)$$

$$= (n-2)\pi - \sum_{i=1}^{n} \alpha_i .$$

$\square$

Notice that by Theorem 25.5, the are of a hyperbolic triangle only depends on its angles, not on the lengths of its sides. In fact, the angles determine the lengths of the sides.

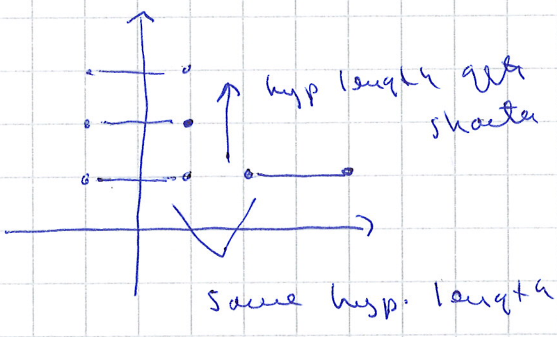
Isometries of $\mathcal{H}$

There are isometries of different types in $\mathcal{H}$:

1) Translations parallel to the real axis:

$T_a(z) = z + a$, $a \in \mathbb{R}$

$(u,v) \mapsto (u+a, v)$

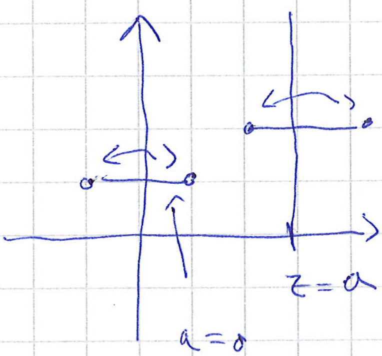


2) Reflections in lines parallel to the imaginary axis:

$R_a(z) = 2a - \bar{z}$, $a \in \mathbb{R}$

$x+iy \mapsto 2a - x + iy$
$= (2a-x) + iy$

$|x-a| = |(2a-x)-a|$



points on the line $z=a$ stays fixed $\qquad (u,v) \mapsto (2a-u, v)$

3) Dilations by $a>0$: $D_a(z) = az$

$(u,v) \mapsto (au, av)$

Each of these 1, 2, 3 preserves the 1st fund form

& preserves the 1st fund form: $f \circ x$ and x have the same 1st fund form