Geometry of Curves and Surfaces, HW 9

1. a) Let S be the cylinder $x^2+y^2=R^2$, $R>0$.

Claim. S is flat $(K=0)$ but not minimal $(H \neq 0)$.

proof. Parametrize S by $X(u,v) = (R\cos u, R\sin u, v)$.
Then

$$X_u = (-R\sin u, R\cos u, 0)$$
$$X_v = (0,0,1)$$
$$X_u \times X_v = \begin{vmatrix} i & j & k \\ -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos u, R\sin u, 0)$$

$\Rightarrow$ unit normal vector is $N_X = \dfrac{X_u \times X_v}{\|X_u \times X_v\|} = (\cos u, \sin u, 0)$

$$E = X_u \cdot X_u = R^2$$
$$F = X_u \cdot X_v = 0$$
$$G = X_v \cdot X_v = 1$$

$$X_{uu} = (-R\cos u, -R\sin u, 0)$$
$$X_{uv} = (0,0,0)$$
$$X_{vv} = (0,0,0)$$

$$L = X_{uu} \cdot N_X = -R$$
$$M = X_{uv} \cdot N_X = 0$$
$$N = X_{vv} \cdot N_X = 0$$

Thus $K = \dfrac{LN - M^2}{EG - F^2} = \dfrac{0-0}{R^2} = 0$, $\therefore$ S is flat

Also, $H = \dfrac{GL + EN - 2FM}{2(EG - F^2)} = \dfrac{-R+0+0}{2(R^2-0)} = \dfrac{-1}{2R} \neq 0$

$\therefore$ S is not minimal

$\square$

k) Calculate the Gaussian and the mean curvatures of the helicoid $X(u,v) = (u\cos v, u\sin v, kv)$, where $k \neq 0$.

$X_u = (\cos v, \sin v, 0)$
$X_v = (-u\sin v, u\cos v, k)$
$X_{uu} = (0,0,0)$
$X_{uv} = (-\sin v, \cos v, 0)$
$X_{vv} = (-u\cos v, -u\sin v, 0)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & k \end{vmatrix} = (k\sin v, -k\cos v, u)$$

$\|X_u \times X_v\| = (k^2 + u^2)^{1/2}$

Unit normal vector $N_x = \dfrac{X_u \times X_v}{\|X_u \times X_v\|} = \dfrac{(k\sin v, -k\cos v, u)}{(k^2+u^2)^{1/2}}$

$E = X_u \cdot X_u = 1 \qquad F = X_u \cdot X_v = 0 \qquad G = X_v \cdot X_v = u^2 + k^2$

$L = X_{uu} \cdot N_x = 0 \qquad N = X_{vv} \cdot N_x = 0$

$M = X_{uv} \cdot N_x = (-k\sin^2 v - k\cos^2 v + 0) \cdot (k^2+u^2)^{1/2} = \dfrac{-k}{(k^2+u^2)^{1/2}}$

$K = \dfrac{LN - M^2}{EG - F^2} = \left(0 - \dfrac{k^2}{k^2+u^2}\right) \cdot (u^2 + k^2 - 0)^{-1} = \dfrac{-k^2}{(k^2+u^2)^2}$

$H = \dfrac{GL + EN - 2FM}{2(EG - F^2)} = \dfrac{0 + 0 + 0}{2(\quad)} = 0.$

2) Calculate the Gaussian and mean curvatures of the saddle surface $z = xy$.

$X(u,v) = (u, v, uv)$

$X_u = (1,0,v)$ $\quad X_v = (0,1,u)$ $\quad X_{uu} = (0,0,0) = X_{vv}$ $\quad X_{uv} = (0,0,1)$

$$X_u \times X_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$$

$$\| X_u \times X_v \| = (v^2+u^2+1)^{1/2} \Rightarrow N_x = \frac{(-v,-u,1)}{(u^2+v^2+1)^{1/2}}.$$

$E = X_u \cdot X_u = 1+v^2$ $\quad F = X_u \cdot X_v = uv$ $\quad G = X_v \cdot X_v = 1+u^2$

$L = X_{uu} \cdot N_x = 0$ $\quad N = X_{vv} \cdot N_x = 0$ $\quad M = X_{uv} \cdot N_x = \frac{1}{(u^2+v^2+1)^{1/2}}$

Thus $$K = \frac{LN-M^2}{EG-F^2} = \left(0 - \frac{1}{u^2+v^2+1}\right) \cdot \frac{1}{1+u^2+v^2+v^2u^2-u^2v^2} = \underline{\frac{-1}{(1+u^2+v^2)^2}}$$

and $$H = \frac{GL+EN-2FM}{2(EG-F^2)} = \left(0+0-2\cdot\frac{uv}{(u^2+v^2+1)^{1/2}}\right) \cdot \frac{1}{2} \cdot \frac{1}{(u^2+v^2+1)}$$

$$= \underline{\frac{-uv}{(u^2+v^2+1)^{3/2}}}$$

3) Find the Gaussian and mean curvatures of the cone $X(u,v) = (u, v, \sqrt{u^2+v^2})$, where $(u,v) \neq (0,0)$.

$X_u = \left(1, 0, \frac{u}{\sqrt{\ }}\right)$ $\quad X_v = \left(0, 1, \frac{v}{\sqrt{\ }}\right)$

$$X_{uu} = \left(0,0,\frac{\sqrt{\ } - u\cdot\frac{u}{\sqrt{\ }}}{u^2+v^2}\right) = \left(0,0,\frac{u^2+v^2-u^2}{(u^2+v^2)^{3/2}}\right) = \left(0,0,\frac{v^2}{(u^2+v^2)^{3/2}}\right)$$

$$X_{uv} = \left(0,0,\frac{0-u\cdot\frac{v}{\sqrt{\ }}}{u^2+v^2}\right) = \left(0,0,\frac{-uv}{(u^2+v^2)^{3/2}}\right)$$

$$X_{vv} = \left(0,0,\frac{u^2}{(u^2+v^2)^{3/2}}\right)$$

$$X_u \times X_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{\ }} \\ 0 & 1 & \frac{v}{\sqrt{\ }} \end{vmatrix} = \left(\frac{-u}{\sqrt{\ }}, \frac{-v}{\sqrt{\ }}, 1\right)$$

$$\|x_u \times x_v\| = \left(\frac{u^2+v^2}{u^2+v^2} + 1 \right)^{1/2} = \sqrt{2}$$

$$\Rightarrow \text{unit normal vector } N_x = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \left(\frac{-u}{\sqrt{2}\sqrt{\ }}, \frac{-v}{\sqrt{2}\sqrt{\ }}, \frac{1}{\sqrt{2}} \right)$$

$$E = x_u \cdot x_u = 1 + 0 + \frac{u^2}{u^2+v^2} = 1 + \frac{u^2}{u^2+v^2}$$

$$F = x_u \cdot x_v = \left(1, 0, \frac{u}{\sqrt{\ }}\right) \cdot \left(0, 1, \frac{v}{\sqrt{\ }}\right) = \frac{uv}{u^2+v^2}$$

$$G = x_v \cdot x_v = 1 + \frac{v^2}{u^2+v^2}$$

$$\left.\begin{aligned} L &= x_{uu} \cdot N_x = \frac{v^2}{\sqrt{2}(u^2+v^2)^{3/2}} \\ M &= x_{uv} \cdot N_x = \frac{-uv}{\sqrt{2}(u^2+v^2)^{3/2}} \\ N &= x_{vv} \cdot N_x = \frac{u^2}{\sqrt{2}(u^2+v^2)^{3/2}} \end{aligned}\right\} \Rightarrow LN - M^2 = 0 \Rightarrow \underline{K = 0}$$

$$EG - F^2 = \|x_u \times x_v\|^2 = 2$$

$$\underline{H} = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{1}{4}\left(\left(1 + \frac{v^2}{u^2+v^2}\right)\left(\frac{v^2}{\sqrt{2}(u^2+v^2)^{3/2}}\right) + \left(1 + \frac{u^2}{u^2+v^2}\right)\left(\frac{u^2}{\sqrt{2}(u^2+v^2)^{3/2}}\right) - 2\frac{uv}{u^2+v^2} \cdot \frac{-uv}{\sqrt{2}(u^2+v^2)^{3/2}} \right)$$

$$= \frac{1}{4}\left(\frac{v^2+u^2}{\sqrt{2}(u^2+v^2)^{3/2}} + \frac{v^4+u^4}{\sqrt{2}(u^2+v^2)^{5/2}} + \frac{2u^2v^2}{\sqrt{2}(u^2+v^2)^{5/2}} \right)$$

$$\frac{(u^2+v^2)^2}{\sqrt{\ }(\quad)}$$

$$= \frac{1}{4}\left(\frac{v^2+u^2}{\sqrt{2}(u^2+v^2)^{3/2}} + \frac{1}{\sqrt{2}(u^2+v^2)^{1/2}} \right) = \frac{1}{4} \cdot \frac{2}{\sqrt{2}(u^2+v^2)^{1/2}} = \underline{\frac{1}{2\sqrt{2}(u^2+v^2)^{1/2}}}$$

4) Show that the following patches have the same Gaussian curvature:

$x(u,v) = (u\cos v, u\sin v, v)$
$y(u,v) = (u\cos v, u\sin v, \ln u)$

The Gaussian curvature for x was calculated in Exercise 1: it is $K_x = \frac{-1}{(1+u^2)^2}$.

Let's calculate the Gaussian curvature K_y for y:

$y_u = (\cos v, \sin v, \frac{1}{u})$ $\quad y_v = (-u\sin v, u\cos v, 0)$

$y_{uu} = (0, 0, \frac{-1}{u^2})$ $\quad y_{uv} = (-\sin v, \cos v, 0)$ $\quad y_{vv} = (-u\cos v, -u\sin v, 0)$

$$y_u \times y_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & \frac{1}{u} \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-\cos v, -\sin v, u)$$

$\Rightarrow$ unit normal vector $N_y = \frac{(-\cos v, -\sin v, u)}{\sqrt{1+u^2}}$

$E = y_u \cdot y_u = 1 + \frac{1}{u^2}$, $F = y_u \cdot y_v = 0$, $G = y_v \cdot y_v = u^2$

$L = y_{uu} \cdot N_y = \frac{-1}{u\sqrt{1+u^2}}$ $\quad M = y_{uv} \cdot N_y = 0$ $\quad N = y_{vv} \cdot N_y = \frac{u}{\sqrt{1+u^2}}$

Thus $K_y = \frac{LN - M^2}{EG - F^2} = \frac{-1}{1+u^2}(u^2+1)^{-1} = \frac{-1}{(1+u^2)^2} = K_x$. $\square$

5) Let x be a regular coordinate patch, and let $c>0$. Then $y = cx$ is also a regular coord. patch. Let K_x and K_y denote the Gaussian curvatures of x and y, respectively.

Claim: $K_y = \frac{1}{c^2} K_x$.

proof. Let $X = X(u,v)$. Then $Y = Y(u,v) = cX(u,v)$.
Thus

$$Y_u = cX_u, \quad Y_v = cX_v, \quad Y_{uu} = cX_{uu}, \quad Y_{uv} = cX_{uv}, \quad Y_{vv} = cX_{vv}.$$

Unit normal vector for X is $N_X = \dfrac{X_u \times X_v}{\|X_u \times X_v\|}$.

Unit normal vector for Y is $N_Y = \dfrac{Y_u \times Y_v}{\|Y_u \times Y_v\|}$

$$= \frac{c^2(X_u \times X_v)}{c^2\|X_u \times X_v\|} = N_X$$

Thus

$$E_Y = Y_u \cdot Y_u = c^2 X_u \cdot X_u = c^2 E_X$$
$$F_Y = Y_u \cdot Y_v = c^2 X_u \cdot X_v = c^2 F_X$$
$$G_Y = Y_v \cdot Y_v = c^2 X_v \cdot X_v = c^2 G_X$$
$$L_Y = Y_{uu} \cdot N_Y = cX_{uu} \cdot N_X = cL_X$$
$$M_Y = Y_{uv} \cdot N_Y = cX_{uv} \cdot N_X = cM_X$$
$$N_Y = Y_{vv} \cdot N_Y = cX_{vv} \cdot N_X = cN_X.$$

Hence

$$K_Y = \frac{L_Y N_Y - M_Y^2}{E_Y G_Y - F_Y^2} = \frac{c^2(L_X N_X - M_X^2)}{c^4(E_X G_X - F_X^2)} = \frac{1}{c^2} K_X. \quad \square$$