Geometry of Curves and Surfaces HW 8

1. Cone $X(u,v) = (v\cos u, v\sin u, v)$

$X_u = (-v\sin u, v\cos u, 0)$
$X_v = (\cos u, \sin u, 1)$

$E = X_u \cdot X_u = v^2$
$F = X_u \cdot X_v = 0$
$G = X_v \cdot X_v = \cos^2 u + \sin^2 u + 1 = 2$

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = (v\cos u, v\sin u, -v)$$

$$N_X = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{(v\cos u, v\sin u, -v)}{[v^2\cos^2 u + v^2\sin^2 u + v^2]^{1/2}} = \frac{1}{\sqrt{2v^2}}(v\cos u, v\sin u, -v)$$

$X_{uu} = (-v\cos u, -v\sin u, 0)$
$X_{uv} = (-\sin u, \cos u, 0)$
$X_{vv} = (0,0,0)$

$$L = X_{uu} \cdot N_X = \frac{1}{\sqrt{2v^2}}(-v^2\cos^2 u - v^2\sin^2 u + 0) = \frac{-v^2}{\sqrt{2v^2}}$$

$$M = X_{uv} \cdot N_X = \frac{1}{\sqrt{2v^2}}(-v\sin u\cos u + v\cos u\sin u + 0) = 0$$

$$N = X_{vv} \cdot N_X = 0 \qquad h = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{-v^2}{\sqrt{2v^2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$h - kg = \frac{-v^2}{\sqrt{2v^2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - k\begin{pmatrix} v^2 & 0 \\ 0 & 2 \end{pmatrix}$$

Assume $v > 0$. Then $\frac{v^2}{\sqrt{2v^2}} = \frac{v^2}{\sqrt{2}\, v} = \frac{v}{\sqrt{2}}$

Then $h - kg = \begin{pmatrix} -v/\sqrt{2} - kv^2 & 0 \\ 0 & -2k \end{pmatrix}$

$$\det(h-kg) = \left(\frac{v}{\sqrt{2}} + kv^2\right) 2k = 0$$

$$\Leftrightarrow \quad k=0 \quad \text{or} \quad k = \frac{-v}{\sqrt{2}\,v^2} = \frac{-1}{\sqrt{2}\,v}$$

$$v<0: \quad \frac{v^2}{\sqrt{2v^2}} = \frac{v^2}{-v\sqrt{2}} = \frac{-v}{\sqrt{2}}$$

$$h - kg = \begin{pmatrix} \frac{v}{\sqrt{2}} - kv^2 & 0 \\ 0 & -2k \end{pmatrix} \Rightarrow \det(h-kg) = \left(\frac{v}{\sqrt{2}} - kv^2\right)(-2k) = 0$$

$$\Leftrightarrow k=0 \quad \text{or} \quad k = \frac{v}{\sqrt{2}\,v^2} = \frac{1}{\sqrt{2}\,v}$$

principal curvatures at $X(u,v)$:

$k_1 = 0, \quad k_2 = \frac{-1}{\sqrt{2}\,v} \quad$ if $v>0$

$k_1 = 0, \quad k_2 = \frac{1}{\sqrt{2}\,v} \quad$ if $v<0$

2. Catenoid $\quad X(u,v) = (u, \cosh u \cos v, \cosh u \sin v)$

$$X_u = (1, \sinh u \cos v, \sinh u \sin v)$$
$$X_v = (0, -\cosh u \sin v, \cosh u \cos v)$$

$$E = X_u \cdot X_u = 1 + \sinh^2 \cos^2 v + \sinh^2 u \sin^2 v = 1 + \sinh^2 u = \cosh^2 u$$
$$F = X_u \cdot X_v = 0$$
$$G = X_v \cdot X_v = \cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v = \cosh^2 u$$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & \sinh u \cos v & \sinh u \sin v \\ 0 & -\cosh u \sin v & \cosh u \cos v \end{vmatrix}$$

$$= (\sinh u \cosh u, -\cosh u \cos v, -\cosh u \sin v)$$

$$\| X_u \times X_v \| = \left[\sinh^2 u \cosh^2 u + \cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v \right]^{1/2}$$
$$= \Big[\underbrace{(1+\sinh^2 u)}_{\cosh^2 u} \cosh^2 u \Big]^{1/2} = \cosh^2 u$$

Thus
$$N_x = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \left(\frac{\sinh u}{\cosh u}, \frac{-\cos v}{\cosh u}, \frac{-\sin v}{\cosh u} \right).$$

$X_{uu} = (0, \cosh u \cos v, \cosh u \sin v)$
$X_{uv} = (0, -\sinh u \sin v, \sinh u \cos v)$
$X_{vv} = (0, -\cosh u \cos v, -\cosh u \sin v)$

$L = X_{uu} \cdot N_x = -\cos^2 v - \sin^2 v = -1$
$M = X_{uv} \cdot N_x = \frac{\sinh u}{\cosh u} \sin v \cos v - \frac{\sinh u}{\cosh u} \sin v \cos v = 0$
$N = X_{vv} \cdot N_x = \cos^2 v + \sin^2 v = 1$

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \cosh^2 u & 0 \\ 0 & \cosh^2 u \end{pmatrix}$$

$$h = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(h - kg) = \det \begin{pmatrix} -1 - k\cosh^2 u & 0 \\ 0 & 1 - k\cosh^2 u \end{pmatrix}$$

$$= +(k\cosh^2 u + 1)(k\cosh^2 u - 1) = 0$$

$$\Leftrightarrow \quad k = \pm \frac{1}{\cosh^2 u}$$

principal curvatures: $k_1 = \frac{1}{\cosh^2 u}$, $k_2 = \frac{-1}{\cosh^2 u}$.

3) $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$

$N : S \to S^2$, the Gauss map.

Let $F : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto x^2 + y^2 - 1$. $S = F^{-1}(0)$

does not depend on z

$\nabla F = (2x, 2y, 0)$
unit normal vector at $(x,y,z) \in S$ is $\frac{\nabla F}{\|\nabla F\|} = \frac{(x,y,0)}{\sqrt{x^2+y^2}}$
Thus
$$N(S) = \left\{ \frac{(x,y,0)}{\underbrace{\sqrt{x^2+y^2}}_{=1}} \,\middle|\, (x,y,0) \in S \right\} = \{(x,y,0) \mid (x,y,0) \in S\}$$
$$= \text{unit circle } S^1.$$

4) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 + (z+2)^2 = 1\}$
$N =$ the Gauss map

Let $F : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto (x-1)^2 + y^2 + (z+2)^2 - 1$. $\Rightarrow S = F^{-1}(0)$.

$\nabla F = (2(x-1), 2y, 2(z+2))$

$\|\nabla F\| = [4(x-1)^2 + 4y^2 + 4(z+2)^2]^{1/2} = 2[\underbrace{(x-1)^2 + y^2 + (z+2)^2}_{=1}]^{1/2} = 2$

Unit normal vector at $(x,y,z) \in S$ is

$$\frac{\nabla F}{\|\nabla F\|} = (x-1, y, z+2).$$

Thus $N(x,y,z) = (x-1, y, z+2)$ and

$N(S) = \{(x-1, y, z+2) \in \mathbb{R}^3 \mid (x-1)^2 + y^2 + (z+2)^2 = 1\}$

$= S^2$.

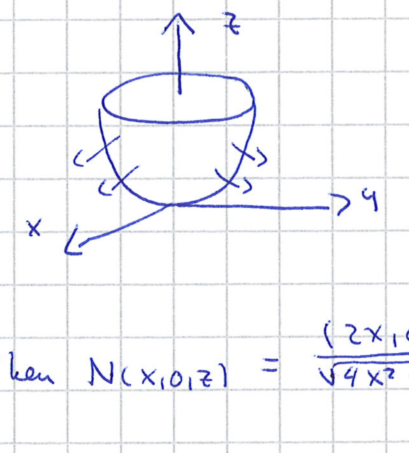
5) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$,
$N =$ the Gauss map.

Let $F : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto x^2 + y^2 - z$. $\Rightarrow S = F^{-1}(0)$.

$\nabla F = (2x, 2y, -1)$

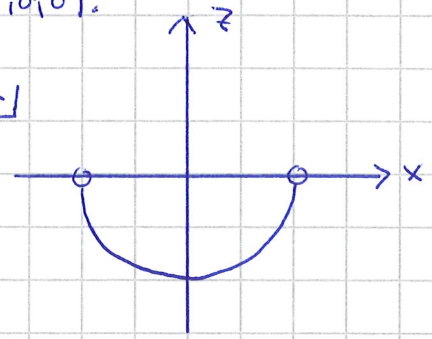
unit normal vector at $(x,y,z) \in S$ is $\frac{\nabla F}{\|\nabla F\|} = \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$

S is a paraboloid:



Let $y = 0$, $(x,0,z) \in S$. Then $N(x,0,z) = \frac{(2x, 0, -1)}{\sqrt{4x^2+1}} \longrightarrow \begin{cases} (1,0,0) & \text{when } x \to \infty \\ (-1,0,0) & \text{when } x \to -\infty \end{cases}$

The image in the xz-plane of the set $\{(x,y,z) \in S \mid y=0\}$ is the half circle connecting $(1,0,0)$ and $(-1,0,0)$.
Notice that here $z<0$.



S is invariant under rotation about the z-axis $\Rightarrow$ also the image $N(S)$ is invariant under the rotation about the z-axis.
Thus

$$N(S) = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ z<0\}.$$

Notice: Different choise (different sign) of the unit normal vector will give the upper hemisphere instead the lower hemisphere.