Geometry of curves and surfaces HW7

1. $X(u,v) = (u, v, u^2+v^2)$

$X_u = (1, 0, 2u) \qquad X_v = (0, 1, 2v)$

$N_X = (X_u \times X_v)/\|X_u \times X_v\|$, where

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

$\|X_u \times X_v\| = [4u^2+4v^2+1]^{1/2} \quad \Rightarrow N_X = [4u^2+4v^2+1]^{-1/2}(-2u, -2v, 1)$

$X_{uu} = (0,0,2), \quad X_{uv} = (0,0,0), \quad X_{vv} = (0,0,2)$

$L = X_{uu} \cdot N_X = 2[4u^2+4v^2+1]^{-1/2}$
$M = X_{uv} \cdot N_X = 0$
$N = X_{vv} \cdot N_X = 2[4u^2+4v^2+1]^{-1/2}$

The second fundamental form is

$$L(du)^2 + 2M\,du\,dv + N(dv)^2 = \frac{2}{[4u^2+4v^2+1]^{1/2}}\left((du)^2+(dv)^2\right).$$

2. Paraboloid S : $X(u,v) = (u, v, u^2+v^2)$
Circle : $\alpha(t) = (\cos t, \sin t, 1)$

Exercise 1 $\Rightarrow$ unit normal for S is $N_X = [4u^2+4v^2+1]^{-1/2}(-2u, -2v, 1)$

$\alpha'(t) = (-\sin t, \cos t, 0) \Rightarrow \|\alpha'(t)\| = 1$, i.e, α is unit-speed

$\alpha''(t) = (-\cos t, -\sin t, 0) \qquad N_X(\alpha(t)) = [4\cos^2 t + 4\sin^2 t + 1]^{-1/2}(-2\cos t, -2\sin t, 1)$
$= 5^{-1/2}(-2\cos t, -2\sin t, 1)$

normal curvature : $k_n = \alpha'' \cdot N_X$
$= 5^{-1/2}(2\cos^2 t + 2\sin^2 t + 0) = \dfrac{2}{\sqrt{5}}$

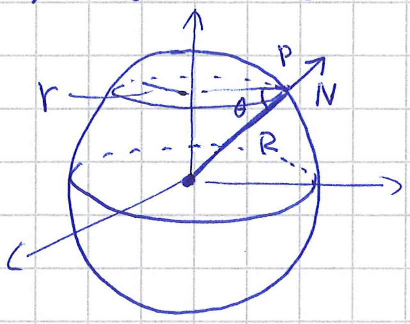
$$N \times (\alpha(t)) \times \alpha' = 5^{-1/2} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2\cos t & -2\sin t & 1 \\ -\sin t & \cos t & 0 \end{vmatrix} = (-\cos t, -\sin t, -2)\, 5^{-1/2}$$

geodesic curvature : $k_g = \alpha'' \cdot (N \times \times \alpha')$

$= (-\cos t, -\sin t, 0) \cdot (-\cos t, -\sin t, -2)\, 5^{-1/2}$

$= 5^{-1/2} = \underline{\underline{\frac{1}{\sqrt{5}}}}$

3) Let S be a sphere of radius R.



α: a circle on S with radius r,

$r = R\cos\theta$

Let p be a point on the circle.
Assume α is parametrized in such a way that it has unit speed.
At p, $n = \alpha''$ is parallel to the line through p perpendicular to the z-axis.

Unit normal N to S is parallel to the line through p and the origin.

Then $\psi = \measuredangle(n, N) = \theta$ or $\pi - \theta$.

Let k = the curvature of α.

The geodesic curvature of α is

$$\underline{\underline{k_g}} = \pm k \sin\psi = \pm \frac{1}{r} \overbrace{\sin\theta}^{= \sin(\pi - \theta)} = \pm \frac{1}{R\cos\theta} \sin\theta = \underline{\underline{\pm \frac{1}{R} \tan\theta.}}$$

4) Let S be the saddle surface $z = x^2 - y^2$.

a) Let $p = (0,0,0)$.

Parametrize S by $X : (u,v) \mapsto (u, v, u^2 - v^2)$

$\Rightarrow X_u = (1, 0, 2u) \qquad X_v = (0, 1, -2v)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = (-2u, 2v, 1)$$

At p, $N = X_u \times X_v = (0,0,1)$, unit normal to S at p.

b) Let $\bar{v} = (1,0,0)$. Then $\bar{v} \in T_pS$.
Let P = the plane determined by N and $\bar{v}$.
Normal vector to P is

$$\bar{v} \times N = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0, -1, 0)$$

The plane P: $0 \cdot (x-0) - 1 \cdot (y-0) + 0 \cdot (z-0) = 0$
$\Leftrightarrow y = 0$.

$P \cap S$: $y = 0 \Rightarrow$ the curve $z = x^2$, call it γ.

Parametrize γ: $\gamma(t) = (t, 0, t^2) \Rightarrow \gamma'(t) = (1, 0, 2t)$
(not unit speed)

$\gamma''(t) = (0,0,2) \qquad \|\gamma'(t)\| = [1+4t^2]^{1/2}$

(At $(0,0,0)$, $t = 0 \Rightarrow \gamma'(0) = (1,0,0)$)

$\|\gamma'(t) \times \gamma''(t)\| = \|(0, -2, 0)\| = 2$

the curvature of γ is $k = \dfrac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \dfrac{2}{[1+4t^2]^{3/2}}$
at $(0,0,0)$, $k = \underline{\underline{2}}$

5) $S: z = x^2 - y^2$, $p = (0,0,0)$

a) $\bar{v} = (0,1,0)$ Exercise 4 $\Rightarrow$ unit normal at p is $N = (0,0,1)$

$\bar{v} \times N = (1,0,0)$, normal vector to P

$P: 1\cdot(x-0) + 0(y-0) + 0(z-0) = 0 \Leftrightarrow x = 0$

$P \cap S: z = -y^2 \Rightarrow$ curve $\gamma(t) = (0, t, -t^2)$

$\Rightarrow \gamma'(t) = (0, 1, -2t) \Rightarrow \|\gamma'(t)\| = [1 + 4t^2]^{1/2}$

$\gamma''(t) = (0,0,-2)$, $\|\gamma'(t) \times \gamma''(t)\| = 2$

the curvature of γ is $k = \dfrac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \dfrac{2}{[1+4t^2]^{3/2}}$

at $(0,0,0)$, $k = 2$

b) $\bar{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $N = (0,0,1)$

$$\bar{v} \times N = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$$

$P: \frac{1}{\sqrt{2}}(x-0) - \frac{1}{\sqrt{2}}(y-0) + 0(z-0) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$

$P \cap S: x = y \Rightarrow z = 0 \Rightarrow \gamma(t) = (t, t, 0)$

$\gamma'(t) = (1,1,0)$

$\gamma''(t) = (0,0,0)$

$\Rightarrow$ the curvature of γ is $\underline{0}$.