Geometry of Curves and Surfaces HW 6

✔

1. $X : \mathbb{R}^2 \to \mathbb{R}, \ (u,v) \mapsto \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$

$$X_u = (1 - u^2 + v^2,\ 2vu,\ 2u)$$
$$X_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$$

$$\begin{aligned} X_u \cdot X_u &= (1-u^2+v^2)^2 + 4v^2u^2 + 4u^2 \\ &= 1 + u^4 + v^4 - 2u^2v^2 - 2u^2 + 2v^2 + 4v^2u^2 + 4u^2 \\ &= 1 + u^4 + v^4 + 2u^2v^2 + 2u^2 + 2v^2 \\ &= (1 + u^2 + v^2)^2 \end{aligned}$$

$$\begin{aligned} X_v \cdot X_v &= 4u^2v^2 + (1-v^2+u^2) + 4v^2 \\ &= (1+u^2+v^2)^2 \end{aligned}$$

$$\begin{aligned} X_u \cdot X_v &= (1-u^2 \overset{+v^2}{)}\, 2uv + 2vu(1-v^2+u^2) + 2u \cdot (-2v) \\ &= 2uv - 2u^3v + 2uv^3 + 2vu - 2v^3u + 2vu^3 - 4uv \\ &= 0 \end{aligned}$$

The first fundamental form of the patch is

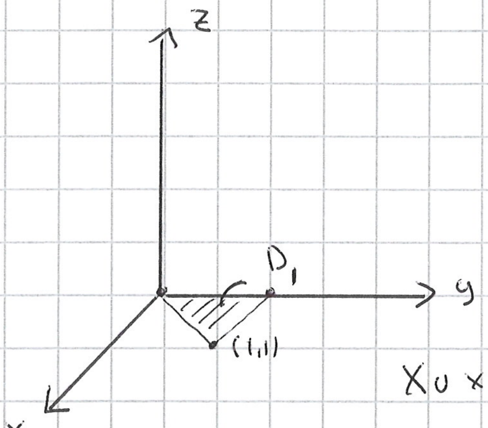
$$(1+u^2+v^2)^2 (du)^2 + (1+u^2+v^2)^2 (dv)^2$$

$$= (1+u^2+v^2)^2 \underbrace{\left[(du)^2 + (dv)^2\right]}_{\text{the first fund. form of the plane}}$$

$\Rightarrow$ The patch is conformal.

2) Find the area of the surface:

a) The part of the plane $z = x + 2y$ that lies above the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$.



parametrize the plane:

$$\mathbf{X}(x,y) = (x, y, x+2y)$$
$$X(u,v) = (u, v, u+2v)$$
$$X_u = (1,0,1) \qquad X_v = (0,1,2)$$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (-1, -2, 1)$$

$$\Rightarrow \|X_u \times X_v\| = \left[(-1)^2 + (-2)^2 + 1^2\right]^{1/2} = \sqrt{6}$$

Let the part of the plane above D be S. Then

$$A(S) = \iint_D \|X_u \times X_v\|\, du\,dv = \iint_D \sqrt{6}\, du\,dv = \sqrt{6} \iint_D du\,dv$$
$$= \sqrt{6}\; A(D) = \sqrt{6} \cdot \frac{1 \cdot 1}{2} = \underline{\underline{\frac{\sqrt{6}}{2}}}$$

b) The part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 4$.

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

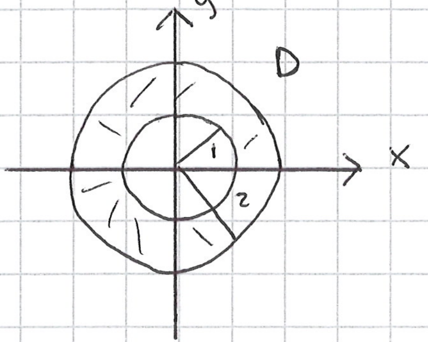
$\Rightarrow$ param. the surface by $X(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

$$X_r = (\cos\theta, \sin\theta, 2r) \qquad X_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$X_r \times X_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\|X_r \times X_\theta\| = [4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2]^{1/2}$$

$$= [4r^4 + r^2]^{1/2}$$



Let the part of the surface $z = x^2+y^2$ between the two cylinders be S.

Then
$$A(S) = \iint_D \|X_r \times X_\theta\|\, dr\, d\theta = \int_1^2\int_0^{2\pi} \sqrt{4r^4+r^2}\, d\theta\, dr$$

$$= 2\pi \int_1^2 r\sqrt{4r^2+1}\, dr = \frac{2\pi}{8}\int_1^2 8r\sqrt{4r^2+1}\, dr$$

$$= \frac{\pi}{4}\left.\frac{(4r^2+1)^{3/2}}{3/2}\right|_1^2 = \frac{2\pi}{12}\left[17^{3/2} - 5^{3/2}\right]$$

$$= \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right].$$

3. Let $S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ z \neq 1,\ z \neq -1\}$
$S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$.

Then
$$X_1(\theta,\varphi) = (\cos\theta\cos\varphi,\ \sin\theta\cos\varphi,\ \sin\varphi)$$
is a patch for S_1, and

$$X_2(\theta,\varphi) = (\cos\theta,\ \sin\theta,\ \sin\varphi)$$
is a patch for S_2.

Let R be a small region in $\mathbb{R}^2$.

<u>Claim.</u> $X_1(R)$ and $X_2(R)$ have the same area.

<u>proof.</u> $A(X_1(R)) = \iint_R \|(X_1)_\theta \times (X_1)_\varphi\|\, d\theta\, d\varphi$, and

$A(X_2(R)) = \iint_R \|(X_2)_\theta \times (X_2)_\varphi\|\, d\theta\, d\varphi$

$(X_1)_\theta = (-\sin\theta\cos\varphi, \cos\theta\cos\varphi, 0)$
$(X_1)_\varphi = (-\cos\theta\sin\varphi, -\sin\theta\sin\varphi, \cos\varphi)$

$E = (X_1)_\theta \cdot (X_1)_\theta = \cos^2\varphi$

$F = (X_1)_\theta \cdot (X_1)_\varphi = \sin\theta\cos\theta\cos\varphi\sin\varphi - \cos\theta\sin\theta\cos\varphi\sin\varphi = 0$

$G = (X_1)_\varphi \cdot (X_1)_\varphi = \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi = 1$

$\Rightarrow \|(X_1)_\theta \times (X_1)_\varphi\| = (EG-F^2)^{1/2} = (\cos^2\varphi)^{1/2} = |\cos\varphi|$

$(X_2)_\theta = (-\sin\theta, \cos\theta, 0)$
$(X_2)_\varphi = (0, 0, \cos\varphi)$

$E = (X_2)_\theta \cdot (X_2)_\theta = \sin^2\theta + \cos^2\theta = 1$

$F = (X_2)_\theta \cdot (X_2)_\varphi = 0$

$G = (X_2)_\varphi \cdot (X_2)_\varphi = \cos^2\varphi$

$\Rightarrow \|(X_2)_\theta \times (X_2)_\varphi\| = (EG-F^2)^{1/2} = (\cos^2\varphi)^{1/2} = |\cos\varphi|$

Since $\|(X_1)_\theta \times (X_1)_\varphi\| = \|(X_2)_\theta \times (X_2)_\varphi\|$, it follows that

$$\mathcal{A}(X_1(R)) = \mathcal{A}(X_2(R)). \quad \square$$

4) Let $C = \{(x,y,z) \in \mathbb{R}^3 \mid 3x^2+3y^2 = z^2, z>0\}$.
Let

$$f : \mathbb{R}^2 - \{0\} \to C$$

$$(r\cos\theta, r\sin\theta) \mapsto \left(\frac{r}{2}\cos 2\theta, \frac{r}{2}\sin 2\theta, \frac{\sqrt{3}}{2}r\right)$$

<u>Claim:</u> f is a local isometry.

proof Let $r > 0$. Then

$$3\left(\frac{r}{2}\cos 2\theta\right)^2 + 3\left(\frac{r}{2}\sin 2\theta\right)^2$$

$$= 3 \cdot \frac{r^2}{4}\left(\cos^2 2\theta + \sin^2 2\theta\right) = \frac{3r^2}{4} = \left(\frac{\sqrt{3}}{2}r\right)^2.$$

Thus $f(r\cos\theta, r\sin\theta) \in C$.

Let $X : (r,\theta) \mapsto (r\cos\theta, r\sin\theta, 0)$ be a patch for $\mathbb{R}^2 - \{0\}$.
Then

$$X_r = (\cos\theta, \sin\theta, 0)$$
$$X_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$\Rightarrow E = X_r \cdot X_r = 1$, $F = X_r \cdot X_\theta = 0$, $G = X_\theta \cdot X_\theta = r^2$

Let $f \circ X : \mathbb{R}^2 - \{0\} \to C$, $(r,\theta) \mapsto \left(\frac{r}{2}\cos 2\theta, \frac{r}{2}\sin 2\theta, \frac{\sqrt{3}}{2}r\right)$.

Then $(f\circ X)_r = \left(\frac{1}{2}\cos 2\theta, \frac{1}{2}\sin 2\theta, \frac{\sqrt{3}}{2}\right)$
$(f\circ X)_\theta = (-r\sin 2\theta, r\cos 2\theta, 0)$

$\Rightarrow E = (f\circ X)_r \cdot (f\circ X)_r = \frac{1}{4} + \frac{3}{4} = 1$
$F = (f\circ X)_r \cdot (f\circ X)_\theta = 0$
$G = (f\circ X)_\theta \cdot (f\circ X)_\theta = r^2$

Thus $(f\circ X)_r \times (f\circ X)_\theta = (EG - F^2)^{1/2} = r \neq 0$

$\Rightarrow$ $f\circ X$ is a local diffeomorphism
$\Rightarrow$ f is a local diffeomorphism.

Since X and $f\circ X$ have the same 1st fundamental form, it follows ~~that~~ f is a local isometry. ∎

5) Let $\alpha(t) = (\cos t, -\frac{1}{4}\sin t, \sin t)$
$\beta(t) = (\cos t, \sin t, \sin 2t)$

At point $(1,0,0)$, $\cos t = 1$. Choose $t = 0$
$\Rightarrow \alpha(0) = (1,0,0) = \beta(0)$.

$\alpha'(t) = (-\sin t, -\frac{1}{4}\cos t, \cos t)$
$\Rightarrow \alpha'(0) = (0, -\frac{1}{4}, 1)$

$\beta'(t) = (-\sin t, \cos t, 2\cos 2t)$
$\Rightarrow \beta'(0) = (0, 1, 2)$

Let θ = the angle of intersection of α and β at $(1,0,0)$.
Then

$$\cos\theta = \frac{\alpha'(0)\cdot\beta'(0)}{\|\alpha'(0)\|\,\|\beta'(0)\|} = \frac{0 - 1/4 + 2}{\sqrt{\frac{1}{16}+1}\,\sqrt{1+4}}$$

$$= \frac{7}{4}\cdot\frac{1}{\sqrt{\frac{17}{16}}\,\sqrt{5}} = \frac{7}{\sqrt{5}\sqrt{17}}$$