Geometry of Curves and Surfaces HW 5

1)

a) $X(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$

$X_u = (\cosh u \sinh v, \cosh u \cosh v, \cosh u)$
$X_v = (\sinh u \cosh v, \sinh u \sinh v, 0)$

$X_u \cdot X_u = \cosh^2 u \sinh^2 v + \cosh^2 u \cosh^2 v + \cosh^2 u$
(E) $= \cosh^2 u (\underbrace{\sinh^2 v + 1}_{\cosh^2 v}) + \cosh^2 u \cosh^2 v = 2\cosh^2 u \cosh^2 v$

$X_u \cdot X_v = \cosh u \sinh u \sinh v \cosh v + \cosh u \sinh u \cosh v \sinh v + 0$
(F) $= 2 \sinh u \cosh u \sinh v \cosh v$

$X_v \cdot X_v = \sinh^2 u \cosh^2 v + \sinh^2 u \sinh^2 v$
(G) $= \sinh^2 u (\underbrace{\cosh^2 v + \sinh^2 v}_{\cosh 2v}) = \sinh^2 u \cosh 2v$

The 1^{st} fund. form is $E(du)^2 + 2F\,du\,dv + G(dv)^2$

$= 2\cosh^2 u \cosh^2 v (du)^2 + 2(2 \sinh u \cosh u \sinh v \cosh v)\,du\,dv$

$+ \sinh^2 u \cosh 2v (dv)^2$

$= 2\cosh^2 u \cosh^2 v (du)^2 + \sinh 2u \sinh 2v\, du\,dv + \sinh^2 u \cosh 2v (dv)^2$

b) $X(u,v) = (u-v, u+v, u^2+v^2)$

$X_u = (1, 1, 2u)$ $\qquad X_v = (-1, 1, 2v)$

$E = X_u \cdot X_u = 1+1+4u^2 = 2+4u^2$
$G = X_v \cdot X_v = (-1)^2 + 1^2 + (2v)^2 = 2+4v^2$
$F = X_u \cdot X_v = -1+1+4uv = 4uv$

1^{st} fund form: $(2+4u^2)(du)^2 + 8uv\,du\,dv + (2+4v^2)(dv)^2$

c) $X(u,v) = (\cosh u, \sinh u, v)$

$X_u = (\sinh u, \cosh u, 0)$ $\quad X_v = (0,0,1)$

$E = X_u \cdot X_u = \sinh^2 u + \cosh^2 u = \cosh 2u$
$F = X_u \cdot X_v = 0+0+0 = 0$
$G = X_v \cdot X_v = 0+0+1^2 = 1$

$E(du)^2 + 2F\,du\,dv + G(dv)^2 = \cosh 2u\,(du)^2 + (dv)^2$

d) $X(u,v) = (u, v, u^2+v^2)$

$X_u = (1, 0, 2u)$ $\quad X_v = (0, 1, 2v)$

$$\left.\begin{aligned} E &= 1^2+0^2+4u^2 = 1+4u^2 \\ F &= 1\cdot 0 + 0\cdot 1 + 2u\cdot 2v = 4uv \\ G &= 0+1^2+4v^2 = 1+4v^2 \end{aligned}\right\} \begin{aligned} &E(du)^2 + 2F(du)(dv) + G(dv)^2 \\ &= (1+4u^2)(du)^2 + (1+4v^2)(dv)^2 \\ &\quad + 8uv\,du\,dv \end{aligned}$$

2) Circle $(x-2)^2 + (y-2)^2 = 1$.
param. by $\alpha(u) = (2+\cos u, 2+\sin u)$

a) Rotate the circle about the x-axis $\Rightarrow$ torus.
Param. the torus:

$X(u,v) = (2+\cos u, (2+\sin u)\cos v, (2+\sin u)\sin v)$

b) $X_u = (-\sin u, \cos u \cos v, \cos u \sin v)$
$X_v = (0, -(2+\sin u)\sin v, (2+\sin u)\cos v)$

$E = X_u \cdot X_u = \sin^2 u + \cos^2 u \cos^2 v + \cos^2 u \sin^2 v = \sin^2 u + \cos^2 u = 1$
$G = X_v \cdot X_v = (2+\sin u)^2 \sin^2 v + (2+\sin u)^2 \cos^2 v$
$= (2+\sin u)^2$
$F = X_u \cdot X_v = 0 - (2+\sin u)\sin v \cos u \cos v + (2+\sin u)\cos v \cos u \sin v$
$= (2+\sin u)(-\sin v \cos u \cos v + \sin v \cos u \cos v) = 0$

$\Rightarrow E(du)^2 + 2F\,du\,dv + G(dv)^2 = (du)^2 + (2+\sin u)^2 (dv)^2$

3) $z = \frac{2xy}{x^2+y^2}$

Monge patch: $(u,v) \mapsto \left(u, v, \frac{2uv}{u^2+v^2}\right)$

Polar coord: $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ patch $(\theta, r) \mapsto \left(r\cos\theta, r\sin\theta, \frac{2r^2\sin\theta\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right)$

$= (r\cos\theta, r\sin\theta, 2\sin\theta\cos\theta)$

$= (0, 0, 2\sin\theta\cos\theta) + r(\cos\theta, \sin\theta, 0)$

$\Rightarrow$ the surface is ruled

4) $z = xy$

patch: $(u,v) \mapsto (u, v, uv) = (u, 0, 0) + v(0, 1, u)$ ruled

Also: $(u,v) \mapsto (u, v, uv) = (0, v, 0) + u(1, 0, v)$ ruled

5) Let $S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = c\}$, where $f: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $\nabla f(x,y,z) \neq 0$ for all $(x,y,z) \in S$.

Claim: ∇f is perpendicular to the tangent plane of S at every point of S.

Proof Let $p = (x_0, y_0, z_0) \in S$ and let $v \in T_pS$. Then there is a smooth curve $\alpha: (a,b) \to S$ and $t_0 \in (a,b)$ s.t. $\alpha'(t_0) = v$. Since $\alpha(t) \in S$ $\forall t \in (a,b)$, it follows that $f(\alpha(t)) = c$ $\forall t \in (a,b)$. Then

$$0 = \frac{d}{dt}(\overset{\text{constant curve } c}{c}) = \frac{d}{dt}(f \circ \alpha)\Big|_{t_0} = \frac{\partial f}{\partial x}\frac{d\alpha_1}{dt}\Big|_{t_0} + \frac{\partial f}{\partial y}\frac{d\alpha_2}{dt}\Big|_{t_0} + \frac{\partial f}{\partial z}\frac{d\alpha_3}{dt}\Big|_{t_0},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

This equals $\underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}_{\nabla f(x_0,y_0,z_0)} \cdot \underbrace{\left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)}_{\alpha'(t_0)}$ at t_0

Then $\nabla f(x_0, y_0, z_0) \cdot \alpha'(t_0) = 0$. Since v was an arbitrary vector in T_pS, it follows that $\nabla f(x_0,y_0,z_0) = \nabla f(p)$ is perpendicular to T_pS. $\square$