# Geometry of Curves and Surfaces HW 4

1) $X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$

$X_u = (1 - u^2 + v^2,\ 2uv,\ 2u)$

$X_v = (2uv,\ 1 - v^2 + u^2,\ -2v)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix}$$

$$= (-4uv^2 - 2u + 2uv^2 - 2u^3,\ 4u^2v + 2v - 2u^2v + 2v^3,$$

$$\underline{1 - v^2 + u^2 - u^2} \underline{+ u^2v^2} - u^4 + v^2 - v^4 \underline{+ u^2v^2} \underline{- 4u^2v^2})$$

$$= (-2uv^2 - 2u - 2u^3,\ 2u^2v + 2v + 2v^3,\ 1 - 2u^2v^2 - u^4 - v^4)$$

$$= (-2u(v^2 + u^2 + 1),\ 2v(u^2 + v^2 + 1),\ 1 - (u^2 + v^2)^2)$$

$u \neq 0$: $-2u(v^2 + u^2 + 1) \neq 0 \Rightarrow X_u \times X_v \neq 0$

$v \neq 0$: $2v(u^2 + v^2 + 1) \neq 0 \Rightarrow X_u \times X_v \neq 0$

$u = v = 0$: $1 - (u^2 + v^2)^2 = 1 \neq 0 \Rightarrow X_u \times X_v \neq 0$.

2)

a) $X(u,v) = (u, v, uv)$

Clearly, $X$ is injective: $X(u,v) = X(u',v')$

$\Rightarrow (u,v,uv) = (u',v',u'v') \Rightarrow u = u',\ v = v'$

$X_u = (1, 0, v) \qquad X_v = (0, 1, u)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1) \quad (\neq \bar{0}\ \forall (u,v))$$

b) $X(u,v) = (u, v^2, v^3)$

$X(u,v) = (u, v^2, v^3) = (u', (v')^2, (v')^3) = X(u', v')$

$\Rightarrow u = u'$ and $\underbrace{v^2 = (v')^2}_{\Rightarrow v = \pm v'}$ and $\underbrace{v^3 = (v')^3}_{\Rightarrow v = v'}$

Then $X$ is injective.

$X_u = (1, 0, 0) \qquad X_v = (0, 2v, 3v^2)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 0 & 2v & 3v^2 \end{vmatrix} = (0, -3v^2, 2v) \quad (\neq 0 \text{ if } v \neq 0)$$

c) $X(u,v) = (u + u^2, v, v^2)$

$X(-1, 0) = (-1 + 1^2, 0, 0) = (0,0,0) = X(0,0) \Rightarrow X$ is not injective

$X_u = (1 + 2u, 0, 0) \qquad X_v = (0, 1, 2v)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1+2u & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = (0, -2v - 4uv, 1 + 2u)$$

$\left( \neq (0,0,0) \text{ if } (u,v) \neq (-\frac{1}{2}, y)\ y \in \mathbb{R} \right.$
i.e. if $\left. u \neq -\frac{1}{2} \right)$

3) Let $x: U = \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$,

$$(u,v) \mapsto \left( \frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \tanh u \right).$$

$x$ is one-to-one:

Let $f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. $(\mathbb{R} \to \mathbb{R})$

Assume $f(x) = f(y)$. Then $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$.

$$\Rightarrow \underbrace{e^{x+y}} + e^{x-y} - e^{-x+y} - \underbrace{e^{-x-y}} = \underbrace{e^{x+y}} - e^{x-y} + e^{-x+y} - \underbrace{e^{-x-y}}$$

$$\Rightarrow 2e^{x-y} = 2e^{-x+y}$$

$$\Rightarrow e^{x-y} = e^{-x+y} \Rightarrow x - y = -x + y \Rightarrow 2x = 2y \Rightarrow x = y.$$

Thus $f$ is injective.

Assume next $x(u,v) = x(\tilde{u}, \tilde{v})$.
Then
$$\left( \frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \tanh u \right) = \left( \frac{\cos \tilde{v}}{\cosh \tilde{u}}, \frac{\sin \tilde{v}}{\cosh \tilde{u}}, \tanh \tilde{u} \right).$$

$f$ injective $\Rightarrow u = \tilde{u} \Rightarrow \begin{cases} \cos v = \cos \tilde{v} \\ \sin v = \sin \tilde{v} \end{cases}$

$\Rightarrow v = \tilde{v}$, since $v, \tilde{v} \in (0, 2\pi)$

$\therefore$ $x$ is one-to-one

Let $x = \frac{\cos v}{\cosh u}$, $y = \frac{\sin v}{\cosh u}$, $z = \tanh u$.

Then
$$x^2 + y^2 + z^2 = \frac{\cos^2 v + \sin^2 v}{\cosh^2 u} + \frac{\sinh^2 u}{\cosh^2 u}$$
$$= \frac{1 + \sinh^2 u}{\cosh^2 u} = \frac{\cosh^2 u}{\cosh^2 u} = 1.$$

Thus $x(U) \subset S^2$.

$$X_u = \left( \frac{-\cos v \sinh u}{\cosh^2 u}, \frac{-\sin v \sinh u}{\cosh^2 u}, \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u} \right)$$

$$= \frac{1}{\cosh^2 u} \left( -\cos v \sinh u, -\sin v \sinh u, 1 \right)$$

$$X_v = \left( \frac{-\sin v}{\cosh u}, \frac{\cos v}{\cosh u}, 0 \right)$$

$$X_u \times X_v = \frac{1}{\cosh^3 u} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\cos v \sinh u & -\sin v \sinh u & 1 \\ -\sin v & \cos v & 0 \end{vmatrix}$$

$$= \frac{1}{\cosh^3 u} \left( -\cos v, -\sin v, -\cos^2 v \sinh u - \sin^2 v \sinh u \right)$$

$$= \frac{1}{\cosh^3 u} \left( -\cos v, -\sin v, -\sinh u \right)$$

Not both 0 at the same time.

$\Rightarrow$ $X_u \times X_v \neq \bar{0}$ at every point in $U$.

4) $X(u,v) = ((2+\sin v)\cos u, (2+\sin v)\sin u, \cos v)$

$X(0,0) = (2\cos(0), 2\sin(0), \cos(0)) = (2,0,1)$

$X_u = ((2+\sin v)(-\sin u), (2+\sin v)\cos u, 0) \Rightarrow X_u(0,0) = (0,2,0)$
$X_v = ((\cos v)\cos u, \cos v \sin u, -\sin v) \Rightarrow X_v(0,0) = (1,0,0)$

$$(X_u \times X_v)(0,0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = (0,0,-2)$$

tangent plane at $(2,0,1)$:

$0(x-2) + 0(y-0) - 2(z-1) = 0$
$\Leftrightarrow$ $2(z-1) = 0$
$\Leftrightarrow$ $\underline{z = 1}$

5)

a) $X(u,v) = (u, v, u^2 - v^2)$ , $(1,1,0)$

$X(u,v) = (1,1,0) \Rightarrow u = 1, v = 1$

$X_u = (1, 0, 2u) \Rightarrow X_u(1,1) = (1,0,2)$

$X_v = (0, 1, -2v) \Rightarrow X_v(1,1) = (0,1,-2)$

$X_u \times X_v =$ (at $(1,1)$) $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} = (-2, 2, 1)$

tangent plane at $(1,1,0)$:

$-2(x-1) + 2(y-1) + 1(z-0) = 0$

$\Leftrightarrow -2x + 2 + 2y - 2 + z = 0$

$\Leftrightarrow \underline{-2x + 2y + z = 0}$

b) $X(r,\theta) = (r\cosh\theta, r\sinh\theta, r^2)$ , $(1,0,1)$

$\left.\begin{array}{l} r\sinh\theta = 0 \Rightarrow r = 0 \text{ or } \theta = 0 \\ r^2 = 1 \Rightarrow r = \pm 1 \end{array}\right\} \Rightarrow \theta = 0$

$r\cosh\theta = 1, \theta = 0 \Rightarrow r = 1$ Thus $(r,\theta) = (1,0)$

$X_r = (\cosh\theta, \sinh\theta, 2r) \Rightarrow X_r(1,0) = (1,0,2)$

$X_\theta = (r\sinh\theta, r\cosh\theta, 0) \Rightarrow X_\theta(1,0) = (0,1,0)$

$(X_r \times X_\theta)(1,0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = (-2, 0, 1)$

tangent plane at $(1,0,1)$:

$-2(x-1) + 0(y-0) + 1(z-1) = 0$

$\Leftrightarrow -2x + 2 + z - 1 = 0$

$\Leftrightarrow \underline{-2x + z + 1 = 0}$