Geometry of Curves and Surfaces HW 3

1) Hyperbolic helix $\alpha(t) = (\cosh t, \sinh t, t)$.

Here $\cosh t = \dfrac{e^t + e^{-t}}{2}$, $\sinh t = \dfrac{e^t - e^{-t}}{2}$.

$\alpha'(t) = (\sinh t, \cosh t, 1)$

$$\Rightarrow |\alpha'(t)| = (\sinh^2 t + \cosh^2 t + 1)^{1/2}$$

$$= \left(\tfrac{1}{4}e^{2t} + \tfrac{1}{4}e^{-2t} - \tfrac{1}{2} + \tfrac{1}{4}e^{2t} + \tfrac{1}{4}e^{-2t} + \tfrac{1}{2} + 1\right)^{1/2}$$

$$= \left(\tfrac{1}{2}e^{2t} + \tfrac{1}{2}e^{-2t} + 1\right)^{1/2} = (\cosh 2t + 1)^{1/2}$$

$\Rightarrow \alpha$ is not a unit speed curve.

$$T = \frac{\alpha'(t)}{|\alpha'(t)|}$$

$$\cosh 2t + 1 = \frac{e^{2t} + e^{-2t}}{2} + 1 = \frac{e^{2t} + e^{-2t} + 2}{2} = \frac{(e^t + e^{-t})^2}{2}$$

$$\Rightarrow |\alpha'(t)| = (\cosh 2t + 1)^{1/2} = \frac{e^t + e^{-t}}{\sqrt{2}} = \sqrt{2}\,\frac{e^t + e^{-t}}{2} = \sqrt{2}\cosh t$$

Thus $$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{(\sinh t, \cosh t, 1)}{\sqrt{2}\cosh t}$$

$$= \frac{1}{\sqrt{2}}\left(\tanh t, 1, \frac{1}{\cosh t}\right).$$

$\alpha'(t) = (\sinh t, \cosh t, 1)$

$\Rightarrow \alpha''(t) = (\cosh t, \sinh t, 0)$

$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix}$$

$$= (-\sinh t, \cosh t, \underbrace{\sinh^2 t - \cosh^2 t}_{-1}) = (-\sinh t, \cosh t, -1).$$

$$|\alpha'(t)\times\alpha''(t)| = (\sinh^2 t + \cosh^2 t + 1)^{1/2} = \sqrt{2}\cosh t$$

like $|\alpha'(t)|$

Thus $B = \dfrac{\alpha'\times\alpha''}{|\alpha'\times\alpha''|} = \dfrac{1}{\sqrt{2}\cosh t}(-\sinh t, \cosh t, -1)$

$$= \frac{1}{\sqrt{2}}\left(-\tanh t, 1, \frac{-1}{\cosh t}\right)$$

$$N = B\times T = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\tanh t & 1 & -1/\cosh t \\ \tanh t & 1 & 1/\cosh t \end{vmatrix}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\left(\frac{2}{\cosh t}, \frac{-\tanh t}{\cosh t}+\frac{\tanh t}{\cosh t}, -\tanh t - \tanh t\right)$$

$$= \left(\frac{1}{\cosh t}, 0, -\tanh t\right)$$

$$k = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3} = \frac{\sqrt{2}\cosh t}{(\sqrt{2}\cosh t)^3} = \frac{1}{2\cosh^2 t}$$

$$(\alpha'\times\alpha'')\cdot\alpha''' = (-\sinh t, \cosh t, -1)\cdot(\sinh t, \cosh t, 0)$$

$$= -\sinh^2 t + \cosh^2 t = 1.$$

$$\tau = \frac{(\alpha'\times\alpha'')\cdot\alpha'''}{|\alpha'\times\alpha''|^2} = \frac{1}{(\sqrt{2}\cosh t)^2} = \frac{1}{2\cosh^2 t}.$$

2) Helix $\alpha(t) = (t+\sqrt{3}\sin t, 2\cos t, \sqrt{3}t - \sin t)$

$\alpha'(t) = (1+\sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t)$
$\alpha''(t) = (-\sqrt{3}\sin t, -2\cos t, \sin t)$
$\alpha'''(t) = (-\sqrt{3}\cos t, 2\sin t, \cos t)$

$$|\alpha'(t)| = \left[(1+\sqrt{3}\cos t)^2 + (-2\sin t)^2 + (\sqrt{3}-\cos t)^2\right]^{1/2}$$

$$= \left[1+3\cos^2 t + \underline{2\sqrt{3}\cos t} + 4\sin^2 t + 3 + \cos^2 t - \underline{2\sqrt{3}\cos t}\right]^{1/2}$$

$$= (4+4)^{1/2} = 2\sqrt{2}$$

$$\alpha' \times \alpha'' = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1+\sqrt{3}\cos t & -2\sin t & \sqrt{3}-\cos t \\ -\sqrt{3}\sin t & -2\cos t & \sin t \end{vmatrix}$$

$$= (-2\sin^2 t + 2\sqrt{3}\cos t - 2\cos^2 t,\ -3\sin t + \sqrt{3}\sin t\cos t$$
$$-\sin t - \sqrt{3}\sin t\cos t,\ -2\cos t - 2\sqrt{3}\cos^2 t - 2\sqrt{3}\sin^2 t)$$

$$= (-2+2\sqrt{3}\cos t,\ -4\sin t,\ -2\cos t - 2\sqrt{3})$$

$$|\alpha' \times \alpha''| = [4 + 12\cos^2 t - 8\sqrt{3}\cos t + 16\sin^2 t + 4\cos^2 t$$
$$+ 12 + 8\sqrt{3}\cos t]^{1/2}$$

$$= [4+16+12]^{1/2} = \sqrt{2 \cdot 16} = 4\sqrt{2}$$

$$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{4\sqrt{2}}{(2\sqrt{2})^3} = \frac{4\sqrt{2}}{8 \cdot 2\sqrt{2}} = \underline{\underline{\frac{1}{4}}}.$$

$$(\alpha' \times \alpha'') \cdot \alpha''' = (-2+2\sqrt{3}\cos t,\ -4\sin t,\ -2\cos t - 2\sqrt{3})$$
$$\cdot (-\sqrt{3}\cos t,\ 2\sin t,\ \cos t)$$

$$= 2\sqrt{3}\cos t - 6\cos^2 t - 8\sin^2 t - 2\cos^2 t - 2\sqrt{3}\cos t = -8$$

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2} = \frac{-8}{(4\sqrt{2})^2} = \frac{-8}{16 \cdot 2} = \underline{\underline{-\frac{1}{4}}}.$$

3) Let $\alpha(t) = (\frac{4}{5}\cos t,\ 1-\sin t,\ -\frac{3}{5}\cos t)$.

Show that $\alpha(t)$ is a circle and find its radius, center and the plane in which it lies.

$$\alpha'(t) = (-\tfrac{4}{5}\sin t,\ -\cos t,\ \tfrac{3}{5}\sin t)$$

$$|\alpha'(t)| = [\tfrac{16}{25}\sin^2 t + \cos^2 t + \tfrac{9}{25}\sin^2 t]^{1/2} = 1 \Rightarrow \text{unit speed}$$

$$\alpha''(t) = (-\tfrac{4}{5}\cos t,\ \sin t,\ \tfrac{3}{5}\cos t)$$

curvature $= k = |\alpha''| = [\frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t]^{1/2} = 1.$

constant

$$N = \frac{\alpha''(t)}{|\alpha''(t)|} = \left(-\frac{4}{5}\cos t,\ \sin t,\ \frac{3}{5}\cos t\right)$$

$$B = T \times N = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \\ -\frac{4}{5}\cos t & \sin t & \frac{3}{5}\cos t \end{vmatrix}$$

$$= \left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t,\ -\frac{12}{25}\sin t\cos t + \frac{12}{25}\sin t\cos t,\right.$$

$$\left.-\frac{4}{5}\sin^2 t - \frac{4}{5}\cos^2 t\right) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right) \Rightarrow B' = 0$$

Torsion $= \tau = -B' \cdot N = 0$.

Thus α is a unit-speed curve with constant curvature and zero torsion. Proposition $\Rightarrow$ α is part of a circle.

Radius $= R$: $k = \frac{1}{R} \Rightarrow R = \frac{1}{k} = \frac{1}{1} = 1$.

Centre : $\alpha(t) + \frac{1}{k}\bar{N} = \left(\frac{4}{5}\cos t,\ 1-\sin t,\ -\frac{3}{5}\cos t\right)$

$$+ \left(-\frac{4}{5}\cos t,\ \sin t,\ \frac{3}{5}\cos t\right) = \underline{(0,1,0)}$$

$\alpha(t)$ lies in the plane that passes through $(0,1,0)$ and contains the vectors T and N. Thus the normal vector of the plane can be chosen to be $B = T \times N$: Equation of the plane

$$-\frac{3}{5}(x-0) + 0(y-1) - \frac{4}{5}(z-0) = 0$$

Equivalently : $3x + 4z = 0$

Notice : When t gets all the real values, $\alpha(t)$ gives the <u>entire</u> circle.

4) $\alpha(t)$ = a smooth unit speed curve with positive curvature.

Darboux vector ω: $T' = \omega \times T$

$N' = \omega \times N$

$B' = \omega \times B$

At any point, $\{T, N, B\}$ is a basis for $\mathbb{R}^3$. $\Rightarrow$ write ω as a linear combination of T, N and B:

$$\omega = aT + bN + cB.$$

Determine a, b, c:

1) $T' = \omega \times T = a\underbrace{T\times T}_{0} + b\underbrace{N\times T}_{-B} + c\underbrace{B\times T}_{N} = -bB + cN \overset{F}{=} kN$

2) $N' = \omega \times N = a\underbrace{T\times N}_{B} + b\underbrace{N\times N}_{0} + c\underbrace{B\times N}_{-T} = aB - cT \overset{F}{=} -kT + \tau B$

3) $B' = \omega \times B = a\underbrace{T\times B}_{-N} + b\underbrace{N\times B}_{T} + c\underbrace{B\times B}_{0} = -aN + bT \overset{F}{=} -\tau N$

$$\left.\begin{array}{l} 1) \Rightarrow b = 0, \quad c = k \\ 2) \Rightarrow a = \tau, \quad c = k \\ 3) \Rightarrow a = \tau, \quad b = 0 \end{array}\right\} \text{consistent!}$$

Thus $\underline{\underline{\omega}} = aT + bN + cB = \underline{\underline{\tau T + kB.}}$

5) Let α, T, N, B and ω be as in exercise 4.

Show that $T' \times T'' = k^2 \omega$, where k is the curvature of α:

Exercise 4 $\Rightarrow$ $\omega = \tau T + kB$, where τ = the torsion of α.

$T' = kN \Rightarrow T'' = (kN)' = k'N + kN' = k'N + k(\omega \times N)$.

$\Rightarrow T' \times T'' = kN \times [k'N + k(\omega \times N)]$

$= (kN) \times (k'N) + kN \times [k(\omega \times N)]$

$= (kk')\underbrace{(N \times N)}_{=0} + k^2[N \times (\omega \times N)]$

$= k^2 N \times [(\underbrace{\tau T + kB}_{\omega}) \times N] = k^2 N \times [\tau \underbrace{T \times N}_{B} + k \underbrace{B \times N}_{-T}]$

$= k^2 N \times [\tau B - kT] = k^2 [\tau \underbrace{N \times B}_{T} - k \underbrace{N \times T}_{-B}]$

$= k^2 [\underbrace{\tau T + kB}_{\omega}] = \underline{\underline{k^2 \omega.}}$ $\square$