Geometry of Curves and Surfaces HW2

1) $\alpha(t) = \left(\frac{(1+t)^{3/2}}{3}, \frac{(1-t)^{3/2}}{3}, \frac{t}{\sqrt{2}} \right), \quad -1 < t < 1.$

a) $\alpha'(t) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right)$

$|\alpha'(t)|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2} = 1 \Rightarrow \alpha$ is a unit-speed curve

$\underline{T = \alpha'(t) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right)}$

$T' = \alpha''(t) = \left(\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0 \right)$

$\Rightarrow$ <u>curvature</u> of α is

$\underline{k} = |T'| = \left[\frac{1}{16}(1+t)^{-1} + \frac{1}{16}(1-t)^{-1} \right]^{1/2} = \frac{1}{\sqrt{16}} \cdot \left(\frac{1}{1+t} + \frac{1}{1-t} \right)^{1/2}$

$= \frac{1}{4} \left(\frac{1-t+1+t}{1-t^2} \right)^{1/2} = \frac{1}{4} \sqrt{\frac{2}{1-t^2}} = \underline{\frac{1}{\sqrt{8(1-t^2)}}}$

$\underline{N} = \frac{T'}{k} = \sqrt{8(1-t^2)} \left(\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0 \right)$

$= \frac{2\sqrt{2}}{4} \left(\sqrt{\frac{1-t^2}{1+t}}, \sqrt{\frac{1-t^2}{1-t}}, 0 \right)$

$= \underline{\frac{\sqrt{2}}{2} \left(\sqrt{1-t}, \sqrt{1+t}, 0 \right)}$

$$\underline{B} = T \times N = \begin{vmatrix} i & j & k \\ \frac{1}{2}\sqrt{1+t} & -\frac{1}{2}\sqrt{1-t} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2(1-t)}}{2} & \frac{\sqrt{2(1+t)}}{2} & 0 \end{vmatrix}$$

$= \left(\frac{-\sqrt{1+t}}{2}, \frac{\sqrt{1-t}}{2}, \frac{\sqrt{2}}{4}(1+t) + \frac{\sqrt{2}}{4}(1-t) \right)$

$= \underline{\left(\frac{-\sqrt{1+t}}{2}, \frac{\sqrt{1-t}}{2}, \frac{\sqrt{2}}{2} \right)} \Rightarrow B' = \left(\frac{-1}{4\sqrt{1+t}}, \frac{-1}{4\sqrt{1-t}}, 0 \right)$

k) The curvature of α was already calculated. The torsion of α is

$$\underline{\tau} = -N \cdot B' = -\left(\frac{\sqrt{2}\sqrt{1-t}}{2}, \frac{\sqrt{2}\sqrt{1+t}}{2}, 0\right) \cdot \left(\frac{-1}{4\sqrt{1+t}}, \frac{-1}{4\sqrt{1-t}}, 0\right)$$

$$= \frac{\sqrt{2}}{8}\frac{\sqrt{1-t}}{\sqrt{1+t}} + \frac{\sqrt{2}}{8}\frac{\sqrt{1+t}}{\sqrt{1-t}} = \frac{\sqrt{2}}{8}\left(\frac{1-t+1+t}{\sqrt{1-t^2}}\right) = \underline{\frac{\sqrt{2}}{4} \cdot \frac{1}{\sqrt{1-t^2}}}$$

2) $\alpha(t) = \left(\frac{1}{\sqrt{2}}\cos t, \sin t, \frac{1}{\sqrt{2}}\cos t\right)$

a) $\alpha'(t) = \left(-\frac{1}{\sqrt{2}}\sin t, \cos t, -\frac{1}{\sqrt{2}}\sin t\right)$

$$|\alpha'(t)|^2 = \frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t = \sin^2 t + \cos^2 t = 1$$

$\Rightarrow$ α is a unit-speed curve and

$$\underline{T = \left(-\frac{1}{\sqrt{2}}\sin t, \cos t, -\frac{1}{\sqrt{2}}\sin t\right)}$$

$$T' = \alpha'' = \left(-\frac{1}{\sqrt{2}}\cos t, -\sin t, -\frac{1}{\sqrt{2}}\cos t\right)$$

$\Rightarrow$ The <u>curvature</u> of α is

$$\underline{k} = |T'| = \left(\frac{1}{2}\cos^2 t + \sin^2 t + \frac{1}{2}\cos^2 t\right)^{1/2} = \underline{1}$$

$$\underline{N} = \frac{1}{k} T' = \underline{\left(-\frac{1}{\sqrt{2}}\cos t, -\sin t, -\frac{1}{\sqrt{2}}\cos t\right)}$$

$$\underline{B} = T \times N = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{1}{\sqrt{2}}\sin t & \cos t & -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\cos t & -\sin t & -\frac{1}{\sqrt{2}}\cos t \end{vmatrix}$$

$$= \left(-\frac{1}{\sqrt{2}}\cos^2 t - \frac{1}{\sqrt{2}}\sin^2 t, \frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos t, \frac{1}{\sqrt{2}}\sin^2 t + \frac{1}{\sqrt{2}}\cos^2 t\right)$$

$$= \underline{\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$$

e) $B' = (0,0,0)$

The torsion of α is

$$\underline{\tau} = -N \cdot B' = \underline{0}.$$

3) $\alpha(t) = \frac{1}{2}\left(t, \frac{1}{t}, \sqrt{2}\ln t\right), \quad t \geq 1$

$$\alpha'(t) = \left(\frac{1}{2}, \frac{-1}{2t^2}, \frac{\sqrt{2}}{2}\cdot\frac{1}{t}\right)$$

$$\alpha''(t) = \left(0, \frac{1}{t^3}, \frac{-\sqrt{2}}{2}\cdot\frac{1}{t^2}\right)$$

$$\alpha' \times \alpha'' = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1/2 & -1/2t^2 & \frac{\sqrt{2}}{2t} \\ 0 & 1/t^3 & \frac{-\sqrt{2}}{2t^2} \end{vmatrix} = \left(\frac{\sqrt{2}}{4t^4} - \frac{\sqrt{2}}{2t^4}, 0 + \frac{\sqrt{2}}{4t^2}, \frac{1}{2t^3}\right)$$

$$= \left(\frac{-\sqrt{2}}{4t^4}, \frac{\sqrt{2}}{4t^2}, \frac{1}{2t^3}\right)$$

$$|\alpha'(t)| = \left(\frac{1}{4} + \frac{1}{4t^4} + \frac{2}{4t^2}\right)^{1/2} = \frac{1}{2}\Big(\underbrace{1 + \left(\frac{1}{t^2}\right)^2 + 2\cdot\frac{1}{t^2}}_{\left(1+\frac{1}{t^2}\right)^2}\Big)^{1/2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{t^2}\right) \qquad (\neq 1)$$

$$|\alpha' \times \alpha''| = \left(\frac{2}{16}\cdot\frac{1}{t^8} + \frac{2}{16}\cdot\frac{1}{t^4} + \frac{1}{4t^6}\right)^{1/2}$$

$$= \frac{\sqrt{2}}{4t^2}\Big(\underbrace{\left(\frac{1}{t^2}\right)^2 + 1 + 2\cdot\frac{1}{t^2}}_{(1+1/t^2)^2}\Big)^{1/2} = \frac{\sqrt{2}}{4t^2}\left(1 + 1/t^2\right)$$

$$\alpha'''(t) = \left(0, \frac{-3}{t^4}, \sqrt{2}\cdot\frac{1}{t^3}\right)$$

$$(\alpha' \times \alpha'')\cdot\alpha''' = \left(\frac{-\sqrt{2}}{4t^2}, \frac{\sqrt{2}}{4t^2}, \frac{1}{2t^3}\right)\cdot\left(0, \frac{-3}{t^4}, \sqrt{2}\cdot\frac{1}{t^3}\right)$$

$$= \frac{-3\sqrt{2}}{4}\cdot\frac{1}{t^6} + \frac{\sqrt{2}}{2}\cdot\frac{1}{t^6} = \frac{-\sqrt{2}}{4}\cdot\frac{1}{t^6}$$

$$\underline{\tau} = \frac{(\alpha' \times \alpha'')\cdot\alpha'''}{|\alpha' \times \alpha''|^2} = \frac{16t^4}{2}\cdot\frac{1}{(1+1/t^2)^2}\cdot\left(\frac{-\sqrt{2}}{4}\cdot\frac{1}{t^6}\right) = \frac{-2\sqrt{2}}{(1+1/t^2)^2 t^2}$$

$$= \underline{\frac{-2\sqrt{2}}{(t + 1/t)^2}}$$

4) Let $a>0$. Let $\alpha(t) = \left(a\cos\frac{t}{c},\ a\sin\frac{t}{c},\ \frac{bt}{c}\right)$,

where $c = \sqrt{a^2+b^2}$.

$$\alpha'(t) = \left(-\frac{a}{c}\sin\frac{t}{c},\ \frac{a}{c}\cos\frac{t}{c},\ \frac{b}{c}\right)$$

$$|\alpha'(t)| = \left(\frac{a^2}{c^2}\sin^2\frac{t}{c} + \frac{a^2}{c^2}\cos^2\frac{t}{c} + \frac{b^2}{c^2}\right)^{1/2}$$

$$= \left(\frac{a^2}{c^2} + \frac{b^2}{c^2}\right)^{1/2} = \left(\frac{a^2+b^2}{a^2+b^2}\right)^{1/2} = 1 \quad \text{unit-speed curve}$$

$$\Rightarrow T(t) = \alpha'(t) = \left(-\frac{a}{c}\sin\frac{t}{c},\ \frac{a}{c}\cos\frac{t}{c},\ \frac{b}{c}\right)$$

$$\Rightarrow T' = \alpha''(t) = \left(-\frac{a}{c^2}\cos\frac{t}{c},\ -\frac{a}{c^2}\sin\frac{t}{c},\ 0\right)$$

The <u>curvature</u> of α is

$$\underline{k} = |T'| = \left(\frac{a^2}{c^4}\cos^2\frac{t}{c} + \frac{a^2}{c^4}\sin^2\frac{t}{c}\right)^{1/2} = \frac{a}{c^2} = \underline{\frac{a}{a^2+b^2}}.$$

$$N = \frac{T'}{k} = \left(-\cos\frac{t}{c},\ -\sin\frac{t}{c},\ 0\right)$$

$$B = T \times N = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{a}{c}\sin\frac{t}{c} & \frac{a}{c}\cos\frac{t}{c} & \frac{b}{c} \\ -\cos\frac{t}{c} & -\sin\frac{t}{c} & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin\frac{t}{c},\ -\frac{b}{c}\cos\frac{t}{c},\ \frac{a}{c}\sin^2\frac{t}{c} + \frac{a}{c}\cos^2\frac{t}{c}\right)$$

$$= \left(\frac{b}{c}\sin\frac{t}{c},\ -\frac{b}{c}\cos\frac{t}{c},\ \frac{a}{c}\right)$$

$$B' = \left(\frac{b}{c^2}\cos\frac{t}{c},\ \frac{b}{c^2}\sin\frac{t}{c},\ 0\right)$$

The torsion of α is

$$\underline{\tau} = -N\cdot B' = \left(\cos\frac{t}{c},\ \sin\frac{t}{c},\ 0\right)\cdot\left(\frac{b}{c^2}\cos\frac{t}{c},\ \frac{b}{c^2}\sin\frac{t}{c},\ 0\right)$$

$$= \frac{b}{c^2} = \underline{\frac{b}{a^2+b^2}}.$$ (Or: Frenet $\Rightarrow B' = -\tau N$ $\Rightarrow \tau = \frac{b}{c^2}$.)

5) $\alpha(t) = (e^t \cos t,\ e^t \sin t,\ e^t)$.

$\alpha'(t) = (e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t)$

$$|\alpha'(t)| = (e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t \cos t + e^{2t})^{1/2}$$

$$= (e^{2t} + e^{2t} + e^{2t})^{1/2} = \sqrt{3}\, e^t$$ (not a unit-speed curve)

Thus $\underline{T(t)} = \frac{\alpha'(t)}{|\alpha'(t)|} = \underline{\frac{1}{\sqrt{3}}(\cos t - \sin t,\ \sin t + \cos t,\ 1)}$

$$\alpha''(t) = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t,\ e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t,\ e^t)$$

$$= (-2e^t \sin t,\ 2e^t \cos t,\ e^t)$$

$$\alpha' \times \alpha'' = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{vmatrix}$$

$$= (e^{2t}\sin t + e^{2t}\cos t - 2e^{2t}\cos t,\ -2e^{2t}\sin t - e^{2t}\cos t + e^{2t}\sin t,\ 2e^{2t}\cos^2 t - 2e^{2t}\sin t \cos t + 2e^{2t}\sin^2 t + 2e^{2t}\sin t \cos t)$$

$$= (e^{2t}\sin t - e^{2t}\cos t,\ -e^{2t}\cos t - e^{2t}\sin t,\ 2e^{2t})$$

$$\Rightarrow |\alpha' \times \alpha''| = e^{2t}(\sin^2 t + \cos^2 t - 2\sin t \cos t + \cos^2 t + \sin^2 t + 2\cos t \sin t + 4)^{1/2}$$

$$= e^{2t}(1+1+4)^{1/2} = \sqrt{6}\, e^{2t}$$

Then $\underline{B} = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = \underline{\frac{1}{\sqrt{6}}(\sin t - \cos t,\ -\cos t - \sin t,\ 2)}$

$$\underline{N} = B \times T$$

$$= \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{3}} \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sin t - \cos t & -\cos t - \sin t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{vmatrix}$$

$$= \frac{1}{3\sqrt{2}} \big(-\cos t - \sin t - 2\sin t - 2\cos t,\ 2\cos t - 2\sin t - \sin t + \cos t,$$
$$\sin^2 t - \cos^2 t + \cos^2 t - \sin^2 t \big)$$

$$= \frac{1}{3\sqrt{2}} \left(-3\cos t - 3\sin t,\ 3\cos t - 3\sin t,\ 0 \right)$$

$$= \underline{\frac{1}{\sqrt{2}} \left(-\cos t - \sin t,\ \cos t - \sin t,\ 0 \right)}$$

$$\underline{k} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{\sqrt{6}\, e^{2t}}{(\sqrt{3}\, e^{t})^3} = \frac{\sqrt{2}\sqrt{3}\, e^{2t}}{3\sqrt{3}\, e^{2t} e^{t}} = \underline{\frac{\sqrt{2}}{3} e^{-t}}$$

$$\alpha''' = \left(-2e^t \sin t - 2e^t \cos t,\ -2e^t \sin t + 2e^t \cos t,\ e^t \right)$$

$$(\alpha' \times \alpha'') \cdot \alpha''' = \left(e^{2t} \sin t - e^{2t} \cos t,\ -e^{2t}\cos t - e^{2t} \sin t,\ 2e^{2t} \right)$$
$$\cdot \left(-2e^t \sin t - 2e^t \cos t,\ -2e^t \sin t + 2e^t \cos t,\ e^t \right)$$

$$= e^{3t} \big(-2\sin^2 t - 2\sin t \cos t + 2\sin t \cos t + 2\cos^2 t$$
$$+ 2\sin t \cos t - 2\cos^2 t + 2\sin^2 t - 2\sin t \cos t + 2 \big)$$

$$= e^{3t} \left(-2 + 2 + 2 \right) = 2e^{3t}$$

$$\underline{\tau} = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2} = \frac{2e^{3t}}{6e^{4t}} = \underline{\frac{1}{3} e^{-t}}.$$