Geometry of Curves and Surfaces, HW 12

1. Let $z, w \in \mathcal{H}$, $\bar{z}$ = the complex conjugate of z
$\bar{w}$ = — " — w.

<u>In class:</u> The hyperbolic distance between z and w is

$$d_{\mathcal{H}}(z,w) = 2\tanh^{-1}\frac{\|z-w\|}{\|z-\bar{w}\|}.$$

<u>Claim:</u>

$$d_{\mathcal{H}}(z,w) = \ln\frac{\|z-\bar{w}\|+\|z-w\|}{\|z-\bar{w}\|-\|z-w\|}.$$

<u>proof.</u> Let $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $(\neq 1)$

$\Rightarrow ye^x + ye^{-x} = e^x - e^{-x} \Rightarrow (y-1)e^x + (y+1)e^{-x} = 0 \quad | \cdot e^x$

$\Rightarrow (y-1)(e^x)^2 + y+1 = 0 \Rightarrow (e^x)^2 = \dfrac{-y-1}{y-1} = \dfrac{y+1}{1-y}$

$\Rightarrow e^x = \sqrt{\dfrac{y+1}{1-y}} \Rightarrow x = \ln\sqrt{\dfrac{y+1}{1-y}} = \dfrac{1}{2}\ln\dfrac{y+1}{1-y}$

Thus $\tanh^{-1}x = \dfrac{1}{2}\ln\dfrac{x+1}{1-x}$

$$\Rightarrow d_{\mathcal{H}}(z,w) = 2\cdot\frac{1}{2}\ln\left(\frac{\frac{\|z-w\|}{\|z-\bar{w}\|}+1}{1-\frac{\|z-w\|}{\|z-\bar{w}\|}}\right)$$

$$= \ln\frac{\|z-w\|+\|z-\bar{w}\|}{\|z-\bar{w}\|-\|z-w\|}. \quad \square$$

2. Let $A = i$, $B = 1+2i$, $C = -1+2i$, $D = 7i$

$$d_{\mathcal{H}}(i, 7i) = \ln\frac{|1-7|+|1+7|}{|1+7|-|1-7|} = \ln\frac{6+8}{8-6} = \underline{\underline{\ln 7}}$$

$$d_{\mathcal{H}}(i, 1+2i) = \ln\frac{\|-1-i\|+\|-1+3i\|}{\|-1+3i\|-\|-1-i\|} = \ln\frac{\sqrt{2}+\sqrt{10}}{\sqrt{10}-\sqrt{2}} = \underline{\underline{\ln\frac{\sqrt{5}+1}{\sqrt{5}-1}}}$$

$$d_{\mathcal{H}}(i,-1+2i) = d_{\mathcal{H}}(i,+1+2i) = \underline{\underline{\ln \frac{\sqrt{5}+1}{\sqrt{5}-1}}}.$$

$$d_{\mathcal{H}}(1+2i,-1+2i) = \ln \frac{\|2\| + \|2+4i\|}{\|2+4i\| - \|2\|} = \ln \frac{\sqrt{20}+\sqrt{4}}{\sqrt{20}-\sqrt{4}} = \underline{\underline{\ln \frac{\sqrt{5}+1}{\sqrt{5}-1}}}.$$

$$d_{\mathcal{H}}(1+2i,7i) = \ln \frac{\|1-5i\| + \|1+9i\|}{\|1+9i\| - \|1-5i\|} = \ln \frac{\sqrt{26}+\sqrt{82}}{\sqrt{82}-\sqrt{26}}$$

$$= \ln \frac{\sqrt{41}+\sqrt{13}}{\sqrt{41}-\sqrt{13}} = \underline{\underline{\ln \frac{41+\sqrt{533}}{41-\sqrt{533}}}}.$$

$$d_{\mathcal{H}}(-1+2i,7i) = d_{\mathcal{H}}(1+2i,7i) = \underline{\underline{\ln \frac{41+\sqrt{533}}{41-\sqrt{533}}}}.$$

3. <u>Claim:</u> There is no positive real number s such that

$$d_{\mathcal{H}}(-s+i,i) = d_{\mathcal{H}}(i,s+i) = d_{\mathcal{H}}(-s+i,s+i).$$

<u>proof.</u> Assume such s exists. Then

$$\frac{\|(-s+i)-i\|}{\|(-s+i)+i\|} = \frac{\|i-(s+i)\|}{\|i-(s-i)\|} = \frac{\|(-s+i)-(s+i)\|}{\|(-s+i)-(s-i)\|}.$$

$$\Rightarrow \frac{s}{\sqrt{s^2+4}} = \frac{s}{\sqrt{s^2+4}} = \frac{2s}{\sqrt{4s^2+4}} = \frac{s}{\sqrt{s^2+1}}.$$

$$\Rightarrow \sqrt{s^2+4} = \sqrt{s^2+1} \Rightarrow s^2+4 = s^2+1 \Rightarrow 4=1 \quad ⚡ \quad \square$$

4. Let $a \in \mathcal{H}$.

<u>Claim:</u> There is a unique hyperbolic line passing through a and intersecting the imaginary axis perpendicularly.

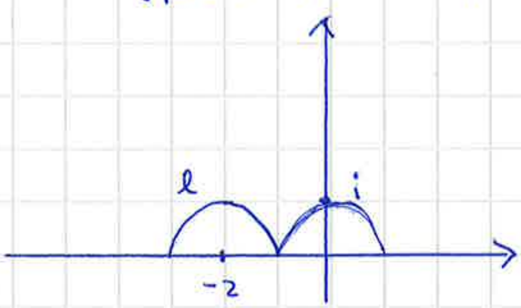
<u>proof.</u> Halflines do not intersect the imaginary axis at all (except the imag. axis, of course). Thus, if such a hyperbolic line exists, it must be a semicircle. Euclidean circles that intersect the imag. axis perpendicularly, are those whose center is on the imag. axis. On the other hand, the center must

be on the real axis, if the semicircle is a hyperbolic line. Therefore, the center must be the origin. But there is exactly one circle passing through a whose center is the origin. This circle is

$$C = \{ z \in \mathbb{C} \mid \|z\| = \|a\| \}.$$

Then the unique hyp. line through a that intersects the imaginary axis perpendicularly, is the semicircle $C \cap \mathcal{H}$. $\square$

5. Let ℓ be the semicircle in $\mathcal{H}$ whose euclidean center is -2 and whose euclidean radius is 1. Then ℓ is a hyperbolic line in $\mathcal{H}$. Find two hyperbolic lines through i that are parallel to ℓ.



Clearly, ℓ_1 = the imag. axis $\cap\ \mathcal{H}$, is a hyperbolic line, and $\ell_1 \cap \ell = \emptyset$.

Let $\ell_2 = \{ z \in \mathcal{H} \mid \|z\| = 1 \}$. Then ℓ_2 is a hyp. line, $i \in \ell_2$ and $\ell \cap \ell_2 = \emptyset$.

Let $\ell_3 = \{ z \in \mathcal{H} \mid \|z+2\| = \|i+2\| = \sqrt{5} \}$. Then ℓ_3 is a hyp. line, $i \in \ell_3$ and $\ell \cap \ell_3 = \emptyset$.