Geometry of Curves and Surfaces HW 11

1. Let $\lambda = \lambda(u,v)$ be a smooth function.
Let $X(u,v)$ be a surface patch having first fundamental form $e^{\lambda}(du^2 + dv^2)$.
Let K be the Gaussian curvature of $X(u,v)$.
Let $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ be the Laplacian.

Claim: $\Delta\lambda + 2Ke^{\lambda} = 0$.

proof: For the patch $X(u,v)$, $F=0$. Then

$$K = \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right).$$

For $X(u,v)$, $E = G = e^{\lambda}$. Thus $G_u = \lambda_u e^{\lambda}$, $E_v = \lambda_v e^{\lambda}$ and $\sqrt{EG} = e^{\lambda}$. Therefore

$$K = -\frac{1}{2}e^{-\lambda}\left(\frac{\partial}{\partial u}\left(e^{-\lambda}\lambda_u e^{\lambda}\right) + \frac{\partial}{\partial v}\left(e^{-\lambda}\lambda_v e^{\lambda}\right)\right)$$

$$= -\frac{1}{2}e^{-\lambda}\left(\frac{\partial^2\lambda}{\partial u^2} + \frac{\partial^2\lambda}{\partial v^2}\right) = -\frac{1}{2}e^{-\lambda}\Delta\lambda. \quad \Big| \cdot 2e^{\lambda}$$

$$\Rightarrow \quad \Delta\lambda + 2Ke^{\lambda} = 0. \ \square$$

2. Claim: There are no surface patches whose first and 2nd fundamental forms are

$$du^2 + (\cos^2 u)\,dv^2 \quad \text{and} \quad (\cos^2 u)\,du^2 + dv^2,$$

respectively.

proof. The Codazzi-Mainardi equations are

$$L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2$$

and

$$M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2.$$

Consider the 2^{nd} C-M equation. Here

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)}$$

$$\Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}$$

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG-F^2)}$$

$$\Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG-F^2)}$$

For the patch: $E=1,\ F=0,\ G=\cos^2 u$
$L = \cos^2 u,\ M=0,\ N=1$

Thus $\Gamma_{22}^1 = \dfrac{-\cos^2 u \cdot (-2\cos u \sin u)}{2\cos^2 u} = \cos u \sin u$

$\Gamma_{22}^2 = 0,\ \Gamma_{12}^1 = 0,\ \Gamma_{12}^2 = \dfrac{-2\cos u \sin u}{2\cos^2 u} = \dfrac{-\sin u}{\cos u}$.

Also, $L = \cos^2 u,\ M=0,\ N=1 \Rightarrow M_v = 0,\ N_v = 0$.

Thus $M_v - N_u = 0$, and

$$L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2 = \cos^2 u \cdot \cos u \sin u + 0 + \frac{\sin u}{\cos u} \neq 0.$$

The 2^{nd} C-M equation does not hold $\Rightarrow$ there can be no surface patch with the given fundamental forms. $\square$

3) Let $x(u,v)$ be a surface patch with the first and second fund. forms

$$E\,du^2 + G\,dv^2 \qquad \text{and} \qquad L\,du^2 + N\,dv^2,$$

respectively.

1. Show that the C-M equations reduce to

$$L_v = \frac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right) \qquad N_u = \frac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right).$$

$M = 0 \Rightarrow$ the C-M equations become

$$L_v = L\Gamma_{12}^1 - N\Gamma_{11}^2$$

and

$$-N_u = L\Gamma_{22}^1 - N\Gamma_{12}^2.$$

Thus $\quad L_v = L\Gamma_{12}^1 - N\Gamma_{11}^2 = L\dfrac{GE_v - FG_u}{2(EG-F^2)} - N\dfrac{2EF_u - EE_v - FE_u}{2(EG-F^2)}$

$$\overset{F=0}{=} \frac{LGE_v}{2EG} + \frac{NEE_v}{2EG} = \frac{1}{2}\left(\frac{LE_v}{E} + \frac{NE_v}{G}\right) = \frac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right)$$

and

$$N_u = -L\Gamma_{22}^1 + N\Gamma_{12}^2 = -L\frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} + N\frac{EG_u - FE_v}{2(EG-F^2)}$$

$$\overset{F=0}{=} -L\cdot\frac{-GG_u}{2EG} + N\frac{EG_u}{2EG} = \frac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right).$$

2. The principal curvatures are $k_1 = \frac{L}{E}$ and $k_2 = \frac{N}{G}$.

Thus $\quad (k_1)_v = \left(\dfrac{L}{E}\right)_v = \dfrac{EL_v - LE_v}{E^2} = \dfrac{L_v}{E} - \dfrac{LE_v}{E^2}$

$$= \frac{E_v}{2E}\left(\frac{L}{E} + \frac{N}{G}\right) - \frac{LE_v}{E^2}$$

$$= \frac{E_v}{2E}\left(-\frac{L}{E} + \frac{N}{G}\right) = \frac{E_v}{2E}(k_2 - k_1),$$

and

$$(k_2)_u = \left(\frac{N}{G}\right)_u = \frac{GN_u - NG_u}{G^2} = \frac{1}{G}\cdot\frac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right) - \frac{NG_u}{G^2}$$

$$= \frac{1}{G}\,\frac{1}{2}\,G_u\left(\frac{L}{E} - \frac{N}{G}\right) = \frac{G_u}{2G}(k_1 - k_2).$$

4. Sphere of radius R: $X(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)\cdot R$

The first fund. form $R^2 d\theta^2 + R^2\cos^2\theta\, d\varphi^2$,
the second fund. form $R\, d\theta^2 + R\cos^2\theta\, d\varphi^2$.

Thus $\quad E = R^2,\ F = 0,\ G = R^2\cos^2\theta$
$\quad L = R,\ M = 0,\ N = R\cos^2\theta$,

and
$$K = \frac{-1}{2\sqrt{EG}}\left(\frac{\partial}{\partial\theta}\left(\frac{G_\theta}{\sqrt{EG}}\right) + \frac{\partial}{\partial\varphi}\left(\frac{E_\varphi}{\sqrt{EG}}\right)\right)$$
$$= \frac{-1}{2\underbrace{\sqrt{R^4(\cos^2\theta)}}_{\substack{R^2\cos\theta \\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}}}}\left(\frac{\partial}{\partial\theta}\left(\frac{-2R^2\cos\theta\sin\theta}{R^2\cos\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{0}{\sqrt{\cdot}}\right)\right)$$
$$= \frac{-1}{2R^2\cos\theta}\frac{\partial}{\partial\theta}(-2\sin\theta) = \frac{1}{R^2\cos\theta}\cos\theta = \underline{\underline{\frac{1}{R^2}}}.$$

5. $X(u,v)$: $E\,du^2 + 2F\,du\,dv + G\,dv^2$
$\quad L\,du^2 + 2M\,du\,dv + N\,dv^2$

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L N_X \qquad X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N N_X$$
$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M N_X$$

$$(X_{uu})_v = (X_{uv})_u \Rightarrow (\Gamma^1_{11}X_u + \Gamma^2_{11}X_v + LN_X)_v = (\Gamma^1_{12}X_u + \Gamma^2_{12}X_v + MN_X)_u$$

$$\Rightarrow \Gamma^1_{11}X_{uv} + (\Gamma^1_{11})_v X_u + \Gamma^2_{11}X_{vv} + (\Gamma^2_{11})_v X_v + L N_{Xv} + L_v N_X$$

$$= \Gamma^1_{12}X_{uu} + (\Gamma^1_{12})_u X_u + \Gamma^2_{12}X_{vu} + (\Gamma^2_{12})_u X_v + M N_{Xu} + M_u N_X$$

Here $\quad N_{Xu} = a X_u + b X_v$
$\quad N_{Xv} = c X_u + d X_v$

Collect the terms containing X_v:

$$\Gamma^1_{11}(\Gamma^2_{12}X_v) + \Gamma^2_{11}(\Gamma^2_{22}X_v) + (\Gamma^2_{11})_v X_v + L d X_v$$
$$(*)$$
$$= \Gamma^1_{12}(\Gamma^2_{11}X_v) + \Gamma^2_{12}(\Gamma^2_{12}X_v) + (\Gamma^2_{12})_u X_v + M b X_v$$

Here $\quad -L = aE + bF \quad |\cdot(-F)$

$-M = cE + dF \quad |\cdot(-F) \quad |\cdot(-G)$

$-N = cF + dG \quad |\cdot E \quad |\cdot F$

$-M = aF + bG \quad |\cdot E$

$$\begin{aligned} FM &= -cEF - dF^2 \\ -EN &= cEF + dEG \\ \hline FM - EN &= d(EG - F^2) \end{aligned} \implies d = \frac{FM-EN}{EG-F^2}$$

$$\begin{aligned} GM &= -cEG - dFG \\ -FN &= cF^2 + dFG \\ \hline GM - FN &= c(F^2 - EG) \end{aligned} \implies c = \frac{FN-GM}{EG-F^2} \Leftarrow$$

not needed!,
we need b

$$\begin{aligned} FL &= -aEF - bF^2 \\ -EM &= aEF + bEG \\ \hline FL - EM &= b(EG - F^2) \end{aligned} \implies b = \frac{FL-EM}{EG-F^2}$$

$$(*) \implies (\Gamma^2_{11})_v - (\Gamma^2_{12})_u = -\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}(-\Gamma^2_{22} + \Gamma^1_{12}) + \Gamma^2_{12}\Gamma^2_{12} + (-Ld + Mb) \qquad (**)$$

Now, $\quad Ld - Mb = \dfrac{FLM - ELN}{EG-F^2} - \dfrac{FLM - EM^2}{EG-F^2} = \dfrac{EM^2 - ELN}{EG-F^2}$

$$= -E \cdot \frac{LN - M^2}{EG-F^2} = -EK.$$

Substitute in $(**)$ $\implies (\Gamma^2_{11})_v - (\Gamma^2_{12})_u = -\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}(-\Gamma^2_{22} + \Gamma^1_{12}) + \Gamma^2_{12}\Gamma^2_{12} + EK$

$$\implies EK = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}(\Gamma^2_{22} - \Gamma^1_{12}) - (\Gamma^2_{12})^2$$