Geometry of Curves and Surfaces, HW 10

1) $S: X(u,v) = (u\cos v,\ u\sin v,\ v)$

a) $\alpha(t) = (a\cos t,\ a\sin t,\ t) = X(a,t),\quad a \in \mathbb{R}$

$X_u = (\cos v, \sin v, 0) \qquad X_v = (-u\sin v,\ u\cos v,\ 1)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v, -\cos v, u)$$

unit normal vector $= N = \dfrac{X_u \times X_v}{\|X_u \times X_v\|} = \dfrac{(\sin v, -\cos v, u)}{\sqrt{1+u^2}} \nearrow \dfrac{(\sin t, -\cos t, a)}{\sqrt{1+a^2}}$.

$\alpha'(t) = (-a\sin t,\ a\cos t,\ 1) \Rightarrow \|\alpha'(t)\| = \sqrt{a^2+1}$

$\alpha''(t) = (-a\cos t,\ -a\sin t,\ 0)$

not needed here

$$\alpha'(t) \times \alpha''(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -a\sin t & a\cos t & 1 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (a\sin t, -a\cos t, a^2)$$

The curvature of α is

$$k(\alpha) = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{(a^2+a^4)^{1/2}}{(a^2+1)^{3/2}} = \frac{|a|(a^2+1)^{1/2}}{(a^2+1)^{3/2}} = \frac{|a|}{a^2+1}.$$

$$N \times \alpha' = (1+a^2)^{-1/2} \cdot \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sin t & -\cos t & a \\ -a\sin t & a\cos t & 1 \end{vmatrix} = (1+a^2)^{-1/2}(-\cos t - a^2\cos t,\ -a^2\sin t - \sin t,\ 0)$$

$$= (1+a^2)^{1/2}(-\cos t, -\sin t, 0)$$

$$\alpha'' \cdot (N \times \alpha') = (-a\cos t, -a\sin t, 0) \cdot (-\cos t, -\sin t, 0)(1+a^2)^{1/2}$$

$$= (a\cos^2 t + a\sin^2 t + 0)(1+a^2)^{1/2} = a(1+a^2)^{1/2}$$

The geodesic curvature of α is

$$k_g(\alpha) = \frac{\alpha'' \cdot (N \times \alpha')}{\|\alpha'\|^3}$$

$$= a(1+a^2)^{1/2} \cdot (a^2+1)^{-3/2}$$

$$= \frac{a}{a^2+1}$$

b) $\beta(t) = (t\cos b, t\sin b, b) = X(t, b)$, where $b \in \mathbb{R}$

$= (0,0,b) + t(\cos b, \sin b, 0)$ param. of a line

$\beta'(t) = (\cos b, \sin b, 0) \Rightarrow \|\beta'(t)\| = (\cos^2 b + \sin^2 b)^{1/2} = 1$

β has unit speed, the geodesic curvature of β is $k_g(\beta) = \beta'' \cdot (N \times \beta') = 0$

2) $S: X(u,v) = (\cos u, \sin u, v) \Rightarrow X_u = (-\sin u, \cos u, 0)$
$X_v = (0,0,1)$

$\alpha(t) = (\cos t, \sin t, a\cos t) = X(t, a\cos t)$

$$X_u \times X_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u, \sin u, 0) = \text{unit normal vector } N$$

$\alpha'(t) = (-\sin t, \cos t, -a\sin t) \Rightarrow \|\alpha'(t)\| = (1 + a^2\sin^2 t)^{1/2}$

$\alpha''(t) = (-\cos t, -\sin t, -a\cos t)$

$N(\alpha(t)) = (\cos t, \sin t, 0)$

$$N \times \alpha' = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & -a\sin t \end{vmatrix} = (-a\sin^2 t, +a\cos t\sin t, 1)$$

$$\alpha'' \cdot (N \times \alpha') = (-\cos t, -\sin t, -a\cos t) \cdot (-a\sin^2 t, a\sin t\cos t, 1)$$

$$= a\sin^2 t\cos t - a\sin^2 t\cos t - a\cos t = -a\cos t$$

The geodesic curvature of α is

$$k_g(\alpha) = \frac{\alpha'' \cdot (N \times \alpha')}{\|\alpha'\|^3} = \frac{-a\cos t}{(1+a^2\sin^2 t)^{3/2}}$$

For the cylinder $X(u,v) = (\cos u, \sin u, v)$

$X_u = (-\sin u, \cos u, 0)$ $\quad X_v = (0,0,1)$

$E = X_u \cdot X_u = 1$, $F = X_u \cdot X_v = 0$, $G = X_v \cdot X_v = 1$

Geodesic equations
$$\begin{cases} \frac{d}{dt}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2) \\ \\ \frac{d}{dt}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2) \end{cases}$$

become
$$\begin{cases} u'' = 0 \\ v'' = 0 \end{cases}.$$

$\alpha(t) = X(t, a\cos t)$, where $u(t) = t$, $v(t) = a\cos t$.

Then $u' = 1 \Rightarrow u'' = 0$, and $v' = -a\sin t \Rightarrow v'' = -a\cos t$.

The curve α is a geodesic iff the geodesic equations are satisfied iff $v'' = -a\cos t = 0$ iff $a = 0$.

3) Let S be as in exercise 2.

Let $\beta(t) = (\cos\omega t, \sin\omega t, at+b) = X(\omega t, at+b)$

$\beta'(t) = (-\omega\sin\omega t, \omega\cos\omega t, a) \Rightarrow \|\beta'(t)\| = (\omega^2 + a^2)^{1/2}$

$\beta''(t) = (-\omega^2\cos\omega t, -\omega^2\sin\omega t, 0)$

$N(\beta(t)) = (\cos\omega t, \sin\omega t, 0)$

$$N \times \beta' = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & a \end{vmatrix} = (a\sin\omega t, -a\cos\omega t, \omega)$$

$$\beta'' \cdot (N \times \beta') = (-\omega^2\cos\omega t, -\omega^2\sin\omega t, 0) \cdot (a\sin\omega t, -a\cos\omega t, \omega)$$

$$= -\omega^2 a\cos\omega t\sin\omega t + a\omega^2\cos\omega t\sin\omega t = 0.$$

$\Rightarrow$ The geodesic curvature of β is $k_g(\beta) = 0$.

As in exercise 2, the geodesic equations become

$$\begin{cases} u'' = 0 \\ v'' = 0 \end{cases}.$$

For β, $u(t) = \omega t$ and $v(t) = at + b$.

Then $u' = \omega \Rightarrow u'' = 0$ and $v' = a \Rightarrow v'' = 0$.

Then β is a geodesic for all values of a, b and ω.

4) Let S be the sphere of radius R and center origin. Then

$$X(u,v) = (R\cos u\cos v, R\sin u\cos v, R\sin v)$$

is a regular coord. patch for S.

$X_u = (-R\sin u\cos v, R\cos u\cos v, 0)$
$X_v = (-R\cos u\sin v, -R\sin u\sin v, R\cos v)$

$E = X_u \cdot X_u = R^2\cos^2 v \Rightarrow E_u = 0$

$G = X_v \cdot X_v = R^2 \Rightarrow G_u = 0$

$F = X_u \cdot X_v = 0$

The geodesic equations

$$\begin{cases} \frac{d}{dt}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2) \\ \frac{d}{dt}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2) \end{cases}$$

become
$$\begin{cases} \frac{d}{dt}(u' R^2 \cos^2 v) = \frac{1}{2}(0+0+0) = 0 \\ \frac{d}{dt}(R^2 v') = \frac{1}{2}(-2R^2 \cos v \sin v)(u')^2 \end{cases}$$

5) $X(u,v)$: E, F, G, L, M, N

<u>Claim</u>: $X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M N_X$, where

N_X = unit normal vector, $\Gamma_{12}^1 = \dfrac{GE_v - FG_u}{2(EG-F^2)}$, $\Gamma_{12}^2 = \dfrac{EG_u - FE_v}{2(EG-F^2)}$.

<u>proof.</u> Write $X_{uv} = \beta_1 X_u + \beta_2 X_v + \beta_3 N_X$

Then $M = X_{uv} \cdot N_X = \beta_3$.

Also

$X_{uv} \cdot X_u = (\beta_1 X_u + \beta_2 X_v + \beta_3 N_X) \cdot X_u = \beta_1 X_u \cdot X_u + \beta_2 X_v \cdot X_u = E\beta_1 + F\beta_2$,

$X_{uv} \cdot X_v = (\beta_1 X_u + \beta_2 X_v + \beta_3 N_X) \cdot X_v = \beta_1 X_u \cdot X_v + \beta_2 X_v \cdot X_v = F\beta_1 + G\beta_2$.

Now, $2 X_{uv} \cdot X_u = (X_u \cdot X_u)_v \Rightarrow X_{uv} \cdot X_u = \frac{1}{2}(X_u \cdot X_u)_v = \frac{1}{2} E_v$

and $X_{uv} \cdot X_v = X_{vu} \cdot X_v = \frac{1}{2}(X_v \cdot X_v)_u = \frac{1}{2} G_u$

Thus
$$\begin{cases} E\beta_1 + F\beta_2 = \frac{1}{2} E_v & | \cdot F & | \cdot G \\ F\beta_1 + G\beta_2 = \frac{1}{2} G_u & | \cdot (-E) & | \cdot (-F) \end{cases}$$

$$\Rightarrow \quad \begin{array}{r} FE\beta_1 + F^2\beta_2 = \frac{1}{2}FE_v \\ -FE\beta_1 - GE\beta_2 = -\frac{1}{2}EG_u \\ \hline (F^2-GE)\beta_2 = \frac{1}{2}(FE_v - EG_u) \end{array}$$

Thus $\quad \Gamma_{12}^2 = \beta_2 = \dfrac{FE_v - EG_u}{2(F^2-GE)} = \dfrac{EG_u - FE_v}{2(EG-F^2)}$.

Similarly, $\quad \begin{array}{r} EG\beta_1 + FG\beta_2 = \frac{1}{2}GE_v \\ -F^2\beta_1 - FG\beta_2 = -\frac{1}{2}FG_u \\ \hline (EG-F^2)\beta_1 = \frac{1}{2}(GE_v - FG_u). \end{array}$

Thus $\quad \Gamma_{12}^1 = \beta_1 = \dfrac{GE_v - FG_u}{2(EG-F^2)}$. $\square$