Geometry of Curves and Surfaces HW 1.

1. $\alpha(t) = (\ln t, \frac{1}{2}t^2, \sqrt{2}\,t)$, $1 \le t \le 2$

$\alpha'(t) = (\frac{1}{t}, t, \sqrt{2})$

$\Rightarrow \|\alpha'(t)\|^2 = \frac{1}{t^2} + t^2 + 2 = (t + \frac{1}{t})^2 \Rightarrow \|\alpha'(t)\| = t + \frac{1}{t}$

The length of α is

$$L(\alpha) = \int_1^2 \|\alpha'(t)\|\,dt = \int_1^2 t + \frac{1}{t}\,dt = \Big|_1^2 \frac{t^2}{2} + \ln t = \frac{4}{2} + \ln 2 - \frac{1}{2} - \ln 1$$
$$= \underline{\underline{\frac{3}{2} + \ln 2}}$$

2. $\alpha(t) = \frac{1}{2}(t, \frac{1}{t}, \sqrt{2}\ln t)$, $t \ge 1$

$\alpha'(t) = \frac{1}{2}(1, -\frac{1}{t^2}, \sqrt{2}\,\frac{1}{t})$

$\Rightarrow \|\alpha'(t)\|^2 = \frac{1}{4}(1 + \frac{1}{t^4} + \frac{2}{t^2}) = \frac{1}{4}(1 + \frac{1}{t^2})^2 \Rightarrow \|\alpha'(t)\| = \frac{1}{2}(1 + \frac{1}{t^2})$

$$S(t) = \int_1^t \|\alpha'(u)\|\,du = \int_1^t \frac{1}{2}(1 + \frac{1}{u^2})\,du = \frac{1}{2}\Big|_1^t u - \frac{1}{u}$$
$$= \frac{1}{2}\left[(t - \frac{1}{t}) - (1 - \frac{1}{1})\right] = \frac{1}{2}(t - \frac{1}{t})$$

$S = \frac{1}{2}(t - \frac{1}{t}) \quad | \cdot 2t$

$\Leftrightarrow 2st = t^2 - 1$

$\Leftrightarrow t^2 - 2st - 1 = 0$

$\Leftrightarrow (t - s)^2 = 1 + s^2$

$\Leftrightarrow t = s \pm \sqrt{1+s^2}$, $t \ge 1$, Thus $t = s + \sqrt{1+s^2}$

$$\alpha(s) = \frac{1}{2}\left(s + \sqrt{1+s^2}, \frac{1}{s+\sqrt{1+s^2}}, \sqrt{2}\ln(s + \sqrt{1+s^2})\right)$$

3.

a) $\alpha(t) = \frac{1}{2}(t, \frac{1}{t}, \sqrt{2}\ln t)$ $\quad t \geq 1$

$$\alpha'(t) = \frac{1}{2}(1, \frac{-1}{t^2}, \sqrt{2}\frac{1}{t}) \Rightarrow \|\alpha'(t)\| = \frac{1}{2}(1 + \frac{1}{t^2})$$

$$\alpha''(t) = \frac{1}{2}(0, \frac{2}{t^3}, -\frac{\sqrt{2}}{t^2})$$

$$\alpha''(t) \times \alpha'(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2/t^3 & -\sqrt{2}/t^2 \\ 1 & -1/t^2 & \sqrt{2}/t \end{vmatrix} \cdot \frac{1}{4}$$

$$= \frac{1}{4}\left(\frac{2\sqrt{2}}{t^4} - \frac{\sqrt{2}}{t^4}, \frac{-\sqrt{2}}{t^2}, -\frac{2}{t^3}\right) = \frac{1}{4t^2}\left(\frac{\sqrt{2}}{t^2}, -\sqrt{2}, -\frac{2}{t}\right)$$

$$\Rightarrow \|\alpha''(t) \times \alpha'(t)\|^2 = \left(\frac{1}{4t^2}\right)^2\left(\frac{2}{t^4} + 2 + \frac{4}{t^2}\right) = \frac{2}{16t^4}\left(\underbrace{\left(\frac{1}{t^2}\right)^2 + 2\cdot\frac{1}{t^2} + 1}_{(1+\frac{1}{t^2})^2}\right)$$

$$= \frac{1}{8t^4}(1 + 1/t^2)^2$$

$$\Rightarrow \|\alpha''(t) \times \alpha'(t)\| = \frac{1}{2\sqrt{2}t^2}(1 + 1/t^2)$$

Thus $\quad k(t) = \frac{\|\alpha''(t) \times \alpha'(t)\|}{\|\alpha'(t)\|^3} = \frac{1}{2\sqrt{2}t^2}\,\frac{(1+1/t^2)}{\frac{1}{8}(1+1/t^2)^3}$

$$= \frac{4}{\sqrt{2}} \cdot \frac{1}{t^2(1+1/t^2)^2} = \underline{\underline{\frac{4}{\sqrt{2}} \cdot \frac{1}{(t+1/t)^2}}}$$

b) $\alpha(t) = \left(\frac{R}{2}\cos\frac{\sqrt{2}}{R}t, \frac{R}{2}\sin\frac{\sqrt{2}}{R}t, \frac{t}{\sqrt{2}}\right)$, $\quad R > 0$

$$\alpha'(t) = \left(-\frac{1}{\sqrt{2}}\sin\frac{\sqrt{2}}{R}t, \frac{1}{\sqrt{2}}\cos\frac{\sqrt{2}}{R}t, \frac{1}{\sqrt{2}}\right)$$

$$\|\alpha'(t)\|^2 = \frac{1}{2}\sin^2\frac{\sqrt{2}}{R}t + \frac{1}{2}\cos^2\frac{\sqrt{2}}{R}t + \frac{1}{2} = 1$$

$$\Rightarrow k(t) = \|\alpha''(t)\|$$

$$\alpha''(t) = \left(-\frac{1}{R}\cos\frac{\sqrt{2}}{R}t,\ -\frac{1}{R}\sin\frac{\sqrt{2}}{R}t,\ 0\right)$$

$$\Rightarrow \|\alpha''(t)\|^2 = \frac{1}{R^2}\cos^2\frac{\sqrt{2}}{R}t + \frac{1}{R^2}\sin^2\frac{\sqrt{2}}{R}t = \frac{1}{R^2}$$

$$\Rightarrow \quad k(t) = \|\alpha''(t)\| = \frac{1}{R}$$

(Notice that the curvature is constant. In fact, it is the same as the curvature of a circle with radius R.)

4.

a) $\alpha(t) = (t, \cosh t)$

$\alpha'(t) = (1, \sinh t) \quad \Rightarrow \|\alpha'(t)\| = \sqrt{1+\sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t$
$\alpha''(t) = (0, \cosh t)$

$$\alpha''(t) \times \alpha'(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & \cosh t & 0 \\ 1 & \sinh t & 0 \end{vmatrix} = (0, 0, -\cosh t)$$

$$\Rightarrow \|\alpha''(t) \times \alpha'(t)\| = \cosh t$$

$$k(t) = \frac{\|\alpha''(t)\times\alpha'(t)\|}{\|\alpha'(t)\|^3} = \frac{\cosh t}{\cosh^3 t} = \frac{1}{\cosh^2 t}$$

b) $\alpha(t) = (\cos^3 t, \sin^3 t)$

$\alpha'(t) = (-3\cos^2 t \sin t,\ 3\sin^2 t \cos t)$

$$\Rightarrow \|\alpha'(t)\|^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t$$
$$= 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) = 9\cos^2 t \sin^2 t$$

$$\alpha''(t) = \left(-6\cos t(-\sin t)\sin t - 3\cos^3 t,\ 6\sin t\cos^2 t - 3\sin^3 t\right)$$

$$\|\alpha'(t)\| = 3\,|\cos t \sin t|$$

$$\alpha''(t)\times\alpha'(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 6\sin^2 t\cos t - 3\cos^3 t & 6\cos^2 t\sin t - 3\sin^3 t & 0 \\ -3\cos^2 t\sin t & 3\sin^2 t\cos t & 0 \end{vmatrix}$$

$$= (0,0,\ 18\sin^4 t\cos^2 t - 9\sin^2 t\cos^4 t + 18\cos^4 t\sin^2 t - 9\cos^2 t\sin^4 t)$$

$$= (0,0,\ 18\sin^2 t\cos^2 t - 9\sin^2 t\cos^2 t)$$

$$= (0,0,\ 9\sin^2 t\cos^2 t)$$

$$\Rightarrow \|\alpha''(t)\times\alpha'(t)\| = 9\sin^2 t\cos^2 t$$

$$k(t) = \frac{\|\alpha''(t)\times\alpha'(t)\|}{\|\alpha'(t)\|^3} = \frac{9\sin^2 t\cos^2 t}{3^3|\cos t\sin t|^3} = (3|\cos t\sin t|)^{-1} = \frac{1}{3|\cos t\sin t|}$$

$t \neq n\cdot\frac{\pi}{2},\ n\in\mathbb{Z}$

c) $\alpha(t) = \left(\frac{4}{5}\cos t,\ 1-\sin t,\ -\frac{3}{5}\cos t\right)$

$$\alpha'(t) = \left(-\frac{4}{5}\sin t,\ -\cos t,\ \frac{3}{5}\sin t\right)$$

$$\|\alpha'(t)\|^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t$$

$$= \sin^2 t + \cos^2 t = 1 \quad \Rightarrow \|\alpha'(t)\| = 1$$

$$\alpha''(t) = \left(-\frac{4}{5}\cos t,\ \sin t,\ \frac{3}{5}\cos t\right)$$

$$\|\alpha''(t)\|^2 = \frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow k(t) = \|\alpha'(t)\| = 1.$$

Notice: Constant curvature one, just like the curvature of a circle with radius 1.

5) $\alpha(t) = \left(\sqrt{\pi}\int_0^t \cos\left(\frac{\pi u^2}{2}\right)du, \sqrt{\pi}\int_0^t \sin\left(\frac{\pi u^2}{2}\right)du\right)$

a) Calculate the curvature of α.

Fundamental theorem of calculus

$$\Rightarrow \alpha'(t) = \left(\sqrt{\pi}\cos\left(\frac{\pi t^2}{2}\right), \sqrt{\pi}\sin\left(\frac{\pi t^2}{2}\right)\right)$$

$$\alpha''(t) = \left(-\sqrt{\pi}\,\frac{2\pi t}{2}\sin\left(\frac{\pi t^2}{2}\right), \sqrt{\pi}\,\frac{2\pi t}{2}\cos\left(\frac{\pi t^2}{2}\right)\right)$$

$$= \left(-\pi^{3/2}t\sin\left(\frac{\pi t^2}{2}\right), \pi^{3/2}t\cos\left(\frac{\pi t^2}{2}\right)\right)$$

$$\|\alpha'(t)\|^2 = \pi\cos^2\left(\frac{\pi t^2}{2}\right) + \pi\sin^2\left(\frac{\pi t^2}{2}\right) = \pi \Rightarrow \|\alpha'(t)\| = \sqrt{\pi}$$

$$\alpha'(t)\times\alpha''(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\frac{\pi t^2}{2} & \sin\frac{\pi t^2}{2} & 0 \\ -t\sin\frac{\pi t^2}{2} & t\cos\frac{\pi t^2}{2} & 0 \end{vmatrix} \cdot \sqrt{\pi} \cdot \pi^{3/2}$$

$$= \pi^2\left(0, 0, t\cos^2\frac{\pi t^2}{2} + t\sin^2\frac{\pi t^2}{2}\right) = (0, 0, \pi^2 t)$$

$$\Rightarrow \|\alpha'(t)\times\alpha''(t)\| = \pi^2 t \quad (t>0)$$

Thus $k(t) = \dfrac{\|\alpha'(t)\times\alpha''(t)\|}{\|\alpha'(t)\|^3} = \dfrac{\pi^2 t}{(\sqrt{\pi})^3} = \underline{\underline{\sqrt{\pi}\,t}}$

b) Calculate the length of the curve from the origin to $\alpha(t)$.

$$L(t) = \int_0^t \|\alpha'(u)\|du \overset{a}{=} \int_0^t \sqrt{\pi}\,du = \sqrt{\pi}\Big|_0^t u = \sqrt{\pi}\,t = k(t)$$

This curve is called <u>Euler's spiral.</u>