GEOMETRY OF CURVES AND SURFACES

Homework 6 (for the week Feb 29 - Mar 4)

(1) Show that the coordinate patch

$$\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^3, \ \ (u,v) \mapsto (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

is a conformal map. (We already had this patch in Homework 4.)

(2) Find the area of the surface:
   (a) The part of the plane $z = x + 2y$ that lies above the triangle with vertices $(0,0)$, $(0,1)$ and $(1,1)$.
   (b) The part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(3) Let

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \neq 1, z \neq -1\}$$

and let

$$S_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, -1 < z < 1\}.$$

Then

$$\mathbf{x}_1(\theta,\varphi) = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$$

is a patch for S_1 and

$$\mathbf{x}_2(\theta,\varphi) = (\cos\theta, \sin\theta, \sin\varphi)$$

is a patch for S_2. Let R be a small region in $\mathbb{R}^2$. Show that the areas of $\mathbf{x}_1(R)$ and $\mathbf{x}_2(R)$ are the same.

(4) Let C be the cone $C = \{(x,y,z) \in \mathbb{R}^3 \mid 3x^2 + 3y^2 = z^2, z > 0\}$. Show that the map $f\colon \mathbb{R}^2 \setminus \{0\} \to C$ defined by

$$f(r\cos\theta, r\sin\theta) = (\frac{r}{2}\cos 2\theta, \frac{r}{2}\sin 2\theta, \frac{\sqrt{3}}{2}r)$$

is a local isometry. (Consider $\mathbb{R}^2 \setminus \{0\}$ as a surface in $\mathbb{R}^3$ by identifying (x,y) with $(x,y,0)$.)

(5) Find the cosine of the angle of intersection of the curves

$$\alpha(t) = (\cos t, -\frac{1}{4}\sin t, \sin t), \ \text{ and } \ \beta(t) = (\cos t, \sin t, \sin 2t)$$

at the point $(1,0,0)$.

Date: February 25, 2016.